Twenty Years of Structural Dynamic Modification – A Review

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Structural dynamic modification techniques have been available as a design tool for several decades. This article reviews the methodology of the technique including both proportional and complex mode formulations. Issues pertaining to limitations on their use and truncation effects of unmeasured modes are discussed. Also discussed is the use of simplistic elements through higher order elements. Discussion on typical commercial implementations is presented. Finally, the ever-constant problems with lack of rotational DOF and truncation are presented, as well as attempts to overcome those issues.

Structural Dynamic Modification (SDM) became a popular modeling tool in the late 1970s following its development and implementation in commercially available software running on desktop computers (albeit predecessors to the IBM PC with limited memory and computation capabilities compared to today's standards). Initial efforts were directed towards implementation of an efficient modeling technique to determine the effect of structural changes using modal data, either test or finite element data, as the basis of the prediction method.

Initial eigenvalue modification techniques required processing matrices that taxed the most powerful of early desktop computers. The introduction of the Local Eigenvalue Modification Procedure (LEMP)[1-3] reduced the more computationally involved eigensolution into a set of second order equations that could easily be handled by almost any computational device. This paved the way to commercial implementation of the Structural Dynamic Modification Process. SDM software[4,5] offered a powerful analytical tool for the test engineer to make quick and simple predictions of the effects of structural changes to a test article before actually implementing the changes. Animation of the modified modes of the system provided a tremendous opportunity to understand the dynamics of the modified system prior to implementing any actual changes.

The first structural elements utilized simple mass, spring and dashpot elements to explore the effects of change to a structural system. Initial approaches to this technique utilized a proportional mode approximation for the solution of system equations. While this approach contained some approximations, the benefit of the computational tool far out-weighed the errors associated with the proportional mode assumption.

Typical studies included effects of simplistic structural changes (mass, damping and stiffness) on predicted frequencies and mode shapes, determination of structural characteristic changes to shift a given resonance, and the effects of the addition of a tuned absorber on the modal characteristics of a system. These tools provided a great improvement to the ability to fine-tune and adjust structural dynamic systems, especially in a troubleshooting environment.

The use of complex modes and the study of damping changes to the system required an enhancement to the equations used for the proportional mode approach. The state space formulation[6,7] for the equations was developed to handle any type of structural change without incurring errors associated with the proportional mode approach. Although the LEMP process was also used for this formulation, many software packages continue to use the proportional mode approach. While very practical engineering solutions can often be obtained from this approach, some solutions are plagued by this approximation.[8]

The LEMP process was the solution scheme of choice until the computational power of desktop computers achieved reasonable speeds. Then the Eigenvalue Modification Technique (EMT) became the typical solution algorithm.[9] Again, these equations can be solved using the proportional mode approximation or by using a full state-space complex mode solution. An assortment of commercially available software packages were developed[10-15] as well as other in-house implementations of the technique.

Once desktop computer systems became more powerful and the EMT became more popular, the introduction of structural elements (typical of the elemental types found in finite element modeling software) became commonplace for the modification of structural systems using modal data as the basis of the computation technique. In addition to component changes on individual components, system models were also developed using this technique[10] but typically were found only on main-

Based on a paper presented at IMAC-XX the 20th International Modal Analysis Conference, Los Angeles, CA, February 2002.

Notation

Matrix

- $[\mathbf{M}]$ = physical mass matrix
- $[\mathbf{C}]$ = physical damping matrix
- $[\mathbf{K}]$ = physical stiffness matrix
- $[\mathbf{A}]$ = physical state space matrix
- $[\mathbf{B}]$ = physical state space matrix
- $[\mathbf{U}]$ = modal matrix – real normal modes
- $[\boldsymbol{\Phi}]$ = modal matrix – complex modes
- $[\mathbf{I}]$ = diagonal modal mass matrix
- $[\boldsymbol{\Lambda}^2]$ = diagonal modal stiffness matrix – unmodified
- $[\boldsymbol{\omega}^2]$ = diagonal modal stiffness matrix – unmodified
- $[\boldsymbol{\Omega}^2]$ = diagonal modal stiffness matrix – modified
- $[\bar{\mathbf{M}}]$ = modal mass matrix
- $[\bar{\mathbf{K}}]$ = modal stiffness matrix

Superscript

- $[\,\bar{}\,]$ = a modal quantity

Vector

- $\{\ddot{\mathbf{x}}\}$ = acceleration
- $\{\dot{\mathbf{x}}\}$ = velocity
- $\{\mathbf{x}\}$ = displacement
- $\{\dot{\mathbf{Y}}\}$ = derivative of state space variable of displacement and velocity
- $\{\mathbf{Y}\}$ = state space variable of displacement and velocity
- $\{\mathbf{F}\}$ = force – real normal mode equations
- $\{\mathbf{Q}\}$ = force – complex mode equations
- $\{\mathbf{p}\}$ = modal displacement
- $\{\mathbf{u}\}$ = modal vector

Subscript

- n = full set of finite element DOF
- a = tested set of experimental DOF
- d = deleted (omitted) set of DOF
- 1 = original set of modes
- 2 = final set of modes
- 12 = transformation from state 1 to state 2

frame computers. As the desktop computer became more powerful, these system modeling tools also found their way into test engineer design software packages.[11,13]

The first useful structural element in the SDM process to be considered was a general 3D beam element (useful for beam and rib types of modification studies).[16,17] However, there was an inherent problem when these realistic structural elements were used in conjunction with modal data obtained from a modal test. The test data lacked the rotational degrees of freedom (RDOF) that are necessary for coupling the elements that contain both translation and rotation information. This lack of rotational DOF was a major obstacle at the introduction of the SDM technique and still presents unique obstacles to efficient implementation. A great deal of research effort was expended in the 1980s to develop techniques for the estimation of rotational DOF[18-22] as well as the development of structural elements that could approximate realistic structural changes by using only the available measured translational DOF.

Many efforts were devoted to the expansion of measured mode shapes for the development of rotational DOF. Some of the techniques utilized shape function approximations[22] while others employed mathematical spline fitting or finite element approximations.[20,21] All of these efforts were undertaken as an intermediate step while an economical rotary accelerometer was developed.

Since rotational measurements may not exist or be available, efforts were focused on approximations of general 3D structural elements using only available translational information. Beam approximations using 3-point bending equations[13,14,23] were the first estimates to be used. These provided reasonably good results for systems that behaved with beam-like responses in the modified system characteristics. In addition, a Guyan reduced beam[11] with only translational DOF also proved to be a very good approximation for systems with beam-like modified characteristics. These two approaches are still used today due to continued difficulties of estimating rotational DOF from measured test data.

However, of all the errors associated with the study of SDM, the most important effect has always been due to the truncation of the modes that describe the modal database.[24-26] The lack of all the modes to adequately describe the dynamic space of the model is by far the most significant error associated with the SDM process. Attempts have been made to estimate residual effects to compensate for some of the out-of-band effects, but these typically are significant only if translational DOF are associated with errors of the model's truncation. However, truncating rotational DOF often causes the most error in all SDM studies and is still a major concern today.

Impedance based techniques were being developed almost parallel to the SDM process and system modeling tools.[27] The modification of a system using only frequency response functions[28] allowed the test engineer to investigate system changes using only measured functions of the unmodified system. The main perceived advantage was that a measurement has no truncation associated with its formulation but rotational DOF have a considerable effect in almost any modeling study. However, impedance based methods only produce frequency response functions – the mode shapes are not a result of the process. Another key drawback is the rarity that rotational DOF are ever available from a modal test. Therefore, any realistic structural changes or system models are always deficient in this regard. Still, the approach has gained significant popularity during the last ten years. Research efforts are directed towards the development of the necessary DOF to produce accurate system models. Implementation of the technique for finite element modeling applications typically results in frequency response information that is affected by truncation of the modes associated with the modeling process. As in the SDM procedures, impedance based modeling techniques are most affected by truncation for analytical approximations and lack of rotational DOF for test applications.

In general, the first ten years of the International Modal Analysis Conference (IMAC) were seen to be the birth and development of the Structural Dynamic Modification Technique. The development of the proportional and complex mode eigenvalue modification technique with the computationally efficient Local Eigenvalue Modification Technique was the subject of many papers in the early years of IMAC. This was followed by the development of more realistic structural elements for component modification studies as well as system models from component modes. The development of tools to estimate rotational DOF was evident during this same period. After the first decade of IMAC, efforts tended to lean towards the utilization of SDM tools and development of system modeling tools. There was also a trend towards using frequency response based tools as a mechanism to develop modifications to components and system models. While the measured FRF may appear to have all the system characteristics (and not be affected by truncation), the effects of rotational DOF (and lack thereof) are important contributors to errors inherent in the results of these modeling techniques. Even today, the two most important errors associated with SDM can often be identified as a lack of rotational DOF to describe the modal database of the system and the truncation of that database.

This article is intended to summarize pertinent development of structural dynamic modeling techniques. Inherent problems that are encountered using the techniques are discussed with appropriate examples to illustrate key concerns and considerations when employing SDM. The general theory of the modification process is presented followed by some of the practical limitations and restrictions that exist in utilizing the techniques.

Theory

A brief theoretical review of the structural modification process is presented herein. Only basic fundamental equations are presented – more in-depth coverage can be found in the references. The basic equations of motion are presented first to identify the nomenclature used. Both the eigenvalue modification and local eigenvalue modification techniques for undamped systems are presented; higher order structural elements are also discussed. This is followed by a general formulation of the equations of motion including arbitrarily damped system using a state space formulation of the equations. All pertinent nomenclature is defined.

Basic Equations of Motion. The equation of motion for a multiple degree of freedom system can be written in matrix form as

$$[\mathbf{M_1}]\{\ddot{\mathbf{x}}\} + [\mathbf{C_1}]\{\dot{\mathbf{x}}\} + [\mathbf{K_1}]\{\mathbf{x}\} = \{\mathbf{F(t)}\} \tag{1}$$

These equations are of size $(n \times n)$ depending on the size of the system matrices. Assuming that the damping matrix is proportional to either the mass or stiffness matrix, the eigen solution at the full space of the physical model can be written as

$$\big[[\mathbf{K_1}] - \lambda[\mathbf{M_1}]\big]\{\mathbf{x}\} = \{\mathbf{0}\} \tag{2}$$

The resulting eigenvalue and eigenvector are noted as ω_1, $\{u_1\}$. The eigenvectors can be arranged in column fashion to form the modal matrix **[U]**. Typically, all of the eigenvectors are not used to describe the system. The modal matrix is therefore rectangular and of size $(n \times m)$.

Using this notation and noting the eigenvalues can be assembled into a diagonal matrix, the eigen problem can be restated as

$$[\mathbf{K_1}][\mathbf{U_1}] = [\mathbf{M_1}][\mathbf{U_1}]\big[\omega_1{}^2\big] \tag{3}$$

Using the modal matrix, a transformation can be made from physical space to modal space using the relationship

$$\{\mathbf{x}\} = [\mathbf{U_1}]\{\mathbf{p_1}\} \tag{4}$$

Substituting this into the equation of motion and pre-multiplying by the transpose of the projection operator to put the equations into normal form gives the standard modal space representation

$$\left[\mathbf{U_1^T}\right][\mathbf{M_1}][\mathbf{U_1}]\{\ddot{\mathbf{p}}_1\}+[\mathbf{U_1}]^{\mathbf{T}}[\mathbf{K_1}][\mathbf{U_1}]\{\mathbf{p_1}\}=[\mathbf{U_1}]^{\mathbf{T}}\{\mathbf{F(t)}\} \quad (5)$$

When the mode shapes are scaled to unit modal mass, then this relationship reduces to

$$[\mathbf{I_1}]\{\ddot{\mathbf{p}}_1\}+\left[\omega_1{}^2\right]\{\mathbf{p_1}\}=\left[\mathbf{U_1^n}\right]^{\mathbf{T}}\{\mathbf{F(t)}\} \quad (6)$$

It is important to note that the diagonal modal mass matrix is

$$[\mathbf{U_1}]^{\mathbf{T}}[\mathrm{M_1}][\mathbf{U_1}]=[\mathbf{I}] \quad (7)$$

and the diagonal modal stiffness matrix is

$$[\mathbf{U_1}]^{\mathbf{T}}[\mathbf{K_1}][\mathbf{U_1}]=\left[\omega_1{}^2\right] \quad (8)$$

Expressing the modal space representation of the system in general terms of the modal mass and modal stiffness gives

$$\begin{bmatrix}\ddots & & \\ & \bar{\mathbf{M}}_1 & \\ & & \ddots\end{bmatrix}\{\ddot{\mathbf{p}}_1\}+\begin{bmatrix}\ddots & & \\ & \bar{\mathbf{K}}_1 & \\ & & \ddots\end{bmatrix}\{\mathbf{p_1}\}=[\mathbf{U_1}]^{\mathbf{T}}\{\mathbf{F(t)}\} \quad (9)$$

(The bar overscore denotes a modal quantity.)

It is very important to note that all of the eigenvalues of the system are generally not available. This implies that the equation size is determined by the number of modes retained in the description of the system. Therefore, these equations are of size ($m \times m$) and are generally much smaller than the physical matrices describing the system. Provided that a sufficient number of modes is retained, not having all the modes does not pose a problem. *However, this often is a problem*. When all the modes of the system are not available, then the system is said to be 'truncated.'

Physical Changes to System Mass and Stiffness. Any changes to the physical system can be written in terms of the modified mass and stiffness as

$$[\mathbf{M_2}]\{\ddot{\mathbf{x}}\}+[\mathbf{K_2}]\{\mathbf{x}\}=\{\mathbf{F(t)}\} \quad (10)$$

The modified mass and stiffness of the system is comprised of the original mass and stiffness plus the change in mass and stiffness as

$$[\mathbf{M_2}]=[\mathbf{M_1}]+[\Delta\mathbf{M_{12}}] \text{ and } [\mathbf{K_2}]=[\mathbf{K_1}]+[\Delta\mathbf{K_{12}}] \quad (11)$$

Another eigensolution at the full space of the physical model can be performed using

$$\left[[\mathbf{K_1}]+[\Delta\mathbf{K_{12}}]\right]-\lambda\left[[\mathbf{M_1}]+[\Delta\mathbf{M_{12}}]\right]\{\mathbf{x}\}=\{\mathbf{0}\} \quad (12)$$

While this eigensolution can be performed, the use of the modal space representation provides numerical efficiencies due to the reduced size of the modal matrices compared with the physical matrices. The physical matrices are typically many orders of magnitude larger those from modal space equations for most engineering applications of structural systems.

Structural Dynamic Modification. Any changes to the physical system can be projected from physical space to modal space through the modal vectors obtained from the original unmodified system. While this modal projection will never produce uncoupled equations, there is tremendous computation benefit due to the significant reduction in size of the modal space equations when compared to the physical set of equations.

Typically once the structural changes are applied in modal space, these modal space equations are fully populated and not diagonal. Two approaches are commonly used for the solution of these equations: the eigenvalue modification (EM) technique and the local eigenvalue modification procedure (LEMP).

Eigenvalue Modification Technique. Considering a change to the system matrices and using the modal space solution as a starting point for the modification, the modification equation can be written as

$$\left[\begin{bmatrix}\ddots & & \\ & \bar{\mathbf{M}}_1 & \\ & & \ddots\end{bmatrix}+\left[\Delta\bar{\mathbf{M}}_{12}\right]\right]\{\ddot{\mathbf{p}}_1\}+\left[\begin{bmatrix}\ddots & & \\ & \bar{\mathbf{K}}_1 & \\ & & \ddots\end{bmatrix}+\left[\Delta\bar{\mathbf{K}}_{12}\right]\right]\{\mathbf{p_1}\}=[\mathbf{0}] \quad (13)$$

where

$$\left[\mathbf{D}\bar{\mathbf{M}}_{12}\right]=[\mathbf{U_1}]^{\mathbf{T}}[\Delta\mathbf{M_{12}}][\mathbf{U_1}]^{\mathbf{T}} \text{ and } \left[\mathbf{D}\bar{\mathbf{K}}_{12}\right]=[\mathbf{U_1}]^{\mathbf{T}}[\Delta\mathbf{K_{12}}][\mathbf{U_1}]^{\mathbf{T}}$$

It is important to note that while part of this equation contains a diagonal representation of the modal mass and modal stiffness of the original system, the remaining change to the modal system is far from diagonal.

Solving for the eigenvalues of this new system using

$$\left[\begin{array}{l}\left[\begin{bmatrix}\ddots & & \\ & \bar{\mathbf{K}}_1 & \\ & & \ddots\end{bmatrix}+[\mathbf{U_1}]^{\mathbf{T}}[\Delta\mathbf{K_{12}}][\mathbf{U_1}]\right] \\ -\lambda\left[\begin{bmatrix}\ddots & & \\ & \bar{\mathbf{M}}_1 & \\ & & \ddots\end{bmatrix}+[\mathbf{U_1}]^{\mathbf{T}}[\Delta\mathbf{M_{12}}][\mathbf{U_1}]\right]\end{array}\right]\{\mathbf{p_1}\}=\{\mathbf{0}\} \quad (14)$$

will yield the new frequencies and another transformation given as

$$\{\mathbf{p_1}\}=[\mathbf{U_{12}}]\{\mathbf{p_2}\} \quad (15)$$

The updated mode shapes can then be obtained from

$$[\mathbf{U_2}]=[\mathbf{U_1}][\mathbf{U_{12}}] \quad (16)$$

This procedure can be used to approximate the modified modal characteristics due to structural changes using only the unmodified modal characteristics of the system. In general, the system will always produce higher frequencies than the exact answer. The more serious the truncation error, the higher the frequency that will be produced from the modification approach. Figure 1 shows the modification process schematically.

When transforming the final modified system to the original physical coordinate system, the relationship of the initial modes to the final modes becomes very apparent through Eq. 16, which provides very revealing information upon expansion of the terms of this matrix. The 'i^{th}' mode, for instance, can be written as

$$\{\mathbf{u_2}\}^{(\mathbf{i})}=\{\mathbf{u_1}\}^{(\mathbf{1})}\mathbf{U_{12}}^{(\mathbf{1i})}+\{\mathbf{u_1}\}^{(\mathbf{2})}\mathbf{U_{12}}^{(\mathbf{2i})}+\{\mathbf{u_1}\}^{(\mathbf{3})}\mathbf{U_{12}}^{(\mathbf{3i})}+\ldots \quad (17)$$

Eq. 17 implies that the final system models are obtained from linear combinations of the starting modes. If a mode that has a significant contribution is not included in $[\mathbf{U_1}]$, then the final system mode in $[\mathbf{U_2}]$ can be seriously in error. This truncation effect is discussed later in more detail.

Local Eigenvalue Modification Technique. The eigenvalue modification technique requires an eigensolution to be performed for matrices that are the size of the number of modes in the modal space solution. The modification equations can be written in a more efficient form using a singular value decomposition of the change matrices in modal space. Without supplying those details here, another formulation of these equations provides tremendous computational advantages and is referred to as the local eigenvalue modification technique. For instance, a general change of the physical system of stiffness can be spectrally decomposed as

$$[\Delta\mathbf{K}]=[\mathbf{T}]\begin{bmatrix}\ddots & & \\ & \alpha & \\ & & \ddots\end{bmatrix}[\mathbf{T}]^{\mathbf{T}}=\sum_{\mathbf{i=1}}^{\mathbf{r}}\{\mathbf{t_i}\}\alpha_{\mathbf{i}}\{\mathbf{t_i}\}^{\mathbf{T}} \quad (18)$$

The importance of this formulation lies in the fact that the singular value decomposition provides vectors that identify the linearly independent pieces that comprise the entire change matrix. These can be written in normal SVD format or as the summation of all the linearly independent pieces that form the change matrix. The later formulation allows for the individual manipulation of each change to the system independently of all other changes.

Assuming that only one type of change (i.e., mass, damping or stiffness) and only one physical change are to occur, then the equations can be re-written into an alternate form. The SVD format of the change to the system can be included in the eigenvalue modification technique. With some manipulation, the resulting equation can be used for the determination of the modified frequencies of the system for stiffness and mass changes, respectively:

$$\frac{1}{\alpha_k} = \sum_{i=1}^{m} \frac{\left(\left\{\mathbf{u}^{(i)}\right\}^T \{\mathbf{t_k}\}\right)^2}{\Omega_2{}^2 - \omega_i{}^2} \tag{19}$$

$$\frac{1}{\Omega^2 \alpha_m} = \sum_{i=1}^{m} \frac{\left(\left\{\mathbf{u}^{(i)}\right\}^T \{\mathbf{t_m}\}\right)^2}{\Omega_2{}^2 - \omega_i{}^2} \tag{20}$$

where the α coefficient results from the singular values of the decomposed change matrix for the stiffness or mass and the $\{t\}$ are the so called "tie vectors" that result from the decomposition for the stiffness or mass.

The LEMP was a very important step in the numerical implementation of the modification equations at a time when computational resources were extremely limited. One advantage of the LEMP is that rather than solving a '$2m$' order set of polynomial equations, the problem was reduced to 'm' second order polynomials that were much easier to solve. Another important feature of the procedure was that the eigenvalues of the modified system were always bounded by the unmodified eigenvalues of the original system. For any numerical procedure used to find the roots of the second order polynomial, the bounds of the root were clearly known, thereby minimizing computational effort. However, the drawback of the LEMP was that only a single modification could be performed. After the modification was made, the system equations had to be updated and additional modifications had to be explored; this tended to be a tedious process. As faster machines became available, the LEMP was eventually replaced by the eigenvalue modification technique.

The LEMP was originally implemented for simple structural elements and was later extended to include higher order elements such as beams, generalized beams and plates. At the time of the LEMP's development, computation capabilities were extremely limited. This procedure provided a viable approach for investigating structural changes to a system using only modal data as the basis of prediction. Today, the LEMP approach is rarely used due to the significantly vast amount of computational resources available even on the slowest of machines.

This technique is widely used in many commercially available software packages. An extension of this method allows for the development of a system model from the component modes of substructures. The basic equations are extended in modal space to address two components, A and B, as

$$\left[\begin{bmatrix} \begin{bmatrix} \ddots & & \\ & \bar{\mathbf{M}}^{\mathbf{A}} & \\ & & \ddots \end{bmatrix} & \\ & \begin{bmatrix} \ddots & & \\ & \bar{\mathbf{M}}^{\mathbf{B}} & \\ & & \ddots \end{bmatrix} \end{bmatrix} + [\mathbf{U}]^{\mathbf{T}}[\Delta\mathbf{M}][\mathbf{U}]\right] \begin{Bmatrix} \{\ddot{\mathbf{p}}^{\mathbf{A}}\} \\ \{\ddot{\mathbf{p}}^{\mathbf{B}}\} \end{Bmatrix}$$
$$+ \left[\begin{bmatrix} \begin{bmatrix} \ddots & & \\ & \bar{\mathbf{K}}^{\mathbf{A}} & \\ & & \ddots \end{bmatrix} & \\ & \begin{bmatrix} \ddots & & \\ & \bar{\mathbf{K}}^{\mathbf{B}} & \\ & & \ddots \end{bmatrix} \end{bmatrix} + [\mathbf{U}]^{\mathbf{T}}[\Delta\mathbf{K}][\mathbf{U}]\right] \begin{Bmatrix} \{\mathbf{p}^{\mathbf{A}}\} \\ \{\mathbf{p}^{\mathbf{B}}\} \end{Bmatrix} = \{\mathbf{0}\} \tag{21}$$

where the modal matrices of the two components, A and B, are written in stacked form as:

$$[\mathbf{U}] = \begin{bmatrix} [\mathbf{U}^{\mathbf{A}}] & \\ & [\mathbf{U}^{\mathbf{B}}] \end{bmatrix} \tag{22}$$

The physical changes to the system are projected to modal space using the modal matrix. The resulting set of equations are then solved to find updated frequencies and mode shapes. This modal modeling approach for SDM and System Modeling can be found in many commercially available software packages.[10-15]

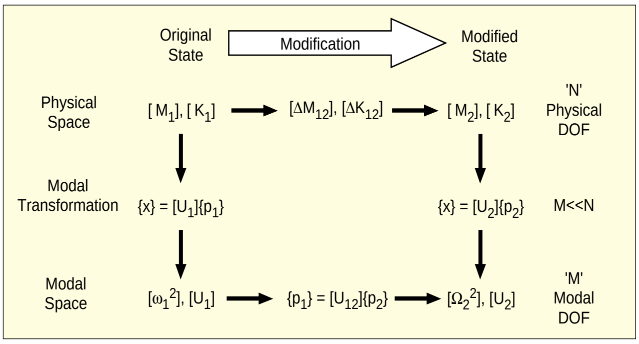

Figure 1. Schematic of the modification process.

Complex Mode Solution. The equations presented above were developed in the absence of damping. In some structural dynamic modification approaches, a proportional model approximation is used for the development of the modification equations. At times, very good modification predictions can be obtained especially when considering only mass and stiffness changes for a proportionally based set of equations. However, when the starting mode shapes are complex in nature or the modifications to be investigated include damping, then the state space solution procedure must be used to ensure accurate results. These equations are presented below to show the basic formulation.

In order to develop the proper set of equations, a supplementary equation is required to describe the equation of motion in state space form. This is presented as

$$\begin{bmatrix} [\mathbf{0}] & [\mathbf{M_1}] \\ [\mathbf{M_1}] & [\mathbf{C_1}] \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{x}} \\ \dot{\mathbf{x}} \end{Bmatrix} - \begin{bmatrix} [\mathbf{M_1}] & [\mathbf{0}] \\ [\mathbf{0}] & [-\mathbf{K_1}] \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{x}} \\ \mathbf{x} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{F} \end{Bmatrix} \tag{23}$$

or

$$[\mathbf{B_1}]\{\dot{\mathbf{Y}}\} - [\mathbf{A_1}]\{\mathbf{Y}\} = \{\mathbf{Q}\} \tag{24}$$

where

$$[\mathbf{B_1}] = \begin{bmatrix} [\mathbf{0}] & [\mathbf{M_1}] \\ [\mathbf{M_1}] & [\mathbf{C_1}] \end{bmatrix} \text{ and } [\mathbf{A_1}] = \begin{bmatrix} [\mathbf{M_1}] & [\mathbf{0}] \\ [\mathbf{0}] & [-\mathbf{K_1}] \end{bmatrix} \tag{25}$$

(Note that some papers and other references use [A] and [B] notation that is interchanged from this notation.)

An eigensolution can be performed for this set of equations

$$\left[[\mathbf{A_1}] - \lambda[\mathbf{B_1}]\right]\{\mathbf{Y}\} = \{\mathbf{0}\} \tag{26}$$

This solution will yield complex frequencies and mode shapes. It is important to note that the matrices are now ($2n \times 2n$) and that the mode shapes obtained are ($2n \times 2$m) and are complex valued.

The modal transformation can be written as:

$$\{\mathbf{Y}\} = [\Phi_1]\{\mathbf{p_1}\} \tag{27}$$

where the mode shapes are complex valued with both displacement and velocity components as

$$\begin{aligned} [\Phi_1] &= \begin{bmatrix} \mathbf{Y^{(1)}} & \vdots & \mathbf{Y^{(2)}} & \vdots & \cdots \end{bmatrix} \\ &= \begin{bmatrix} \{\mathbf{u^{(1)}}\}\ \lambda_1 & \vdots & \{\mathbf{u^{(2)}}\}\ \lambda_2 & \vdots & \cdots \\ \{\mathbf{u^{(1)}}\} & & \{\mathbf{u^{(2)}}\} & & \end{bmatrix} \\ &= \begin{bmatrix} [\mathbf{U_1}] \begin{bmatrix} \ddots & & \\ & \Lambda_1 & \\ & & \ddots \end{bmatrix} \\ [\mathbf{U_1}] \end{bmatrix} \end{aligned} \tag{28}$$

The transformation can be substituted into the equation of motion and premultiplied by the transform of the projection operator (to put the equations into normal form) to obtain

$$[\Phi_1]^{\mathbf{T}}[\mathbf{B_1}][\Phi_1]\{\ddot{\mathbf{p}}_1\} + [\Phi_1]^{\mathbf{T}}[\mathbf{A_1}][\Phi_1]\{\mathbf{p_1}\} = [\Phi_1]^{\mathbf{T}}\{\mathbf{Q(t)}\} \tag{29}$$

which is analogous to the proportional based solution without any damping included. (It is very important to note that if the damping is zero or proportional to the mass and stiffness matrices, these equations will reduce to those previously developed.)

Now the equation of motion in modal space can be written as

$$\begin{bmatrix} \ddots & & \\ & \bar{\mathbf{B}} & \\ & & \ddots \end{bmatrix}\{\dot{\mathbf{p}}\} - \begin{bmatrix} \ddots & & \\ & \bar{\mathbf{A}} & \\ & & \ddots \end{bmatrix}\{\mathbf{p}\} = \left[\bar{\mathbf{Q}}(\mathbf{t})\right] \tag{30}$$

As before, structural changes can now be considered as

$$[\mathbf{M_2}]\{\ddot{\mathbf{x}}\} + [\mathbf{C_2}]\{\dot{\mathbf{x}}\} + [\mathbf{K_2}]\{\mathbf{x}\} = \{\mathbf{F}(\mathbf{t})\} \tag{31}$$

where

$$\begin{aligned} [\mathbf{M_2}] &= [\mathbf{M_1}] + [\Delta\mathbf{M_{12}}] \\ [\mathbf{C_2}] &= [\mathbf{C_1}] + [\Delta\mathbf{C_{12}}] \\ [\mathbf{K_2}] &= [\mathbf{K_1}] + [\Delta\mathbf{K_{12}}] \end{aligned} \tag{32}$$

and must be written in state space form as

$$[\mathbf{B_2}]\{\dot{\mathbf{Y}}\} - [\mathbf{A_2}]\{\mathbf{Y}\} = \{\mathbf{Q}(\mathbf{t})\} \tag{33}$$

where

$$[\mathbf{A_2}] = [\mathbf{A_1}] + [\Delta\mathbf{A_{12}}];\ [\mathbf{B_2}] = [\mathbf{B_1}] + [\Delta\mathbf{B_{12}}] \tag{34}$$

with

$$[\Delta\mathbf{B_{12}}] = \begin{bmatrix} [\mathbf{0}] & [\Delta\mathbf{M_{12}}] \\ [\Delta\mathbf{M_{12}}] & [\Delta\mathbf{C_{12}}] \end{bmatrix} \mathbf{and}\, [\Delta\mathbf{A_{12}}] = \begin{bmatrix} [\Delta\mathbf{M_{12}}] & [\mathbf{0}] \\ [\mathbf{0}] & [-\Delta\mathbf{K_{12}}] \end{bmatrix} \tag{35}$$

This modified equation in modal space can be written as the original unmodified modal quantities plus the change in modal quantities as

$$\left[\begin{bmatrix} \ddots & & \\ & \mathbf{I_1} & \\ & & \ddots \end{bmatrix} + \left[\Delta\bar{\mathbf{B}}_{\mathbf{12}}\right]\right]\{\dot{\mathbf{p}}\} - \left[\begin{bmatrix} \ddots & & \\ & \mathbf{\Lambda_1} & \\ & & \ddots \end{bmatrix} + \left[\Delta\bar{\mathbf{A}}_{\mathbf{12}}\right]\right]\{\mathbf{p}\} = \{\bar{\mathbf{Q}}(\mathbf{t})\} \tag{36}$$

where

$$\begin{aligned} \left[\Delta\bar{\mathbf{A}}_{\mathbf{12}}\right] &= [\mathbf{\Phi_1}]^{\mathbf{T}}[\Delta\mathbf{A_{12}}][\mathbf{\Phi_1}] \\ \left[\Delta\bar{\mathbf{B}}_{\mathbf{12}}\right] &= [\mathbf{\Phi_1}]^{\mathbf{T}}[\Delta\mathbf{B_{12}}][\mathbf{\Phi_1}] \end{aligned} \tag{37}$$

The eigensolution results in the modified system. In the equations above, any changes of mass, damping and stiffness can be handled. In addition, the LEMP can be easily included to add computational efficiences.[7] Again, the LEMP restrictions are that each individual modification needs to be made one at a time.

In this set of equations using the full state space solution, the proper mathematical solution is obtained for arbitrary damping. If the damping is in fact proportional, then these equations reduce to the proportional mode approach. However, it is very important to note that the structural modifications studied can in fact disrupt any proportionality that may exist in the original set of modes. If this is the case, then the full complex mode solution in state space needs to be utilized.

Structural Modification Using Response Functions. An interesting parallel method that uses frequency response functions with an impedance approach is referred to as the Structural Modification Using Response Functions (SMURF) technique.[27] It is presented here as an interesting alternate approach that affords some better understanding of the results obtained from the SDM approach and is included for completeness.

For reference, recall that the frequency responses are made up of the mode shapes of the system depending on the particular point to point frequency response function considered using

$$H_{ij}(j\omega) = \sum_{k=1}^{m} \frac{q_k u_{ik} u_{jk}}{(j\omega - p_k)} + \frac{q_k u_{ik}{}^* u_{jk}{}^*}{(j\omega - p_k{}^*)} \tag{38}$$

Consider the beam shown in Figure 2. For a cantilever beam,

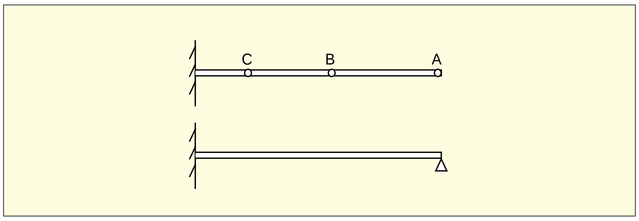

Figure 2. Cantilever beam example.

the frequency response at H_{cb} when $x_a = 0$ is desired; this corresponds to a pinned beam at the end of the cantilever. The response at 'a' is related to the force at 'a' and 'b' through

$$x_a = H_{ab}F_b + H_{aa}F_a \tag{39}$$

With the constraint $x_a = 0$, the force at 'a' is

$$F_a = -H_{aa}^{-1}H_{ab}F_b \tag{40}$$

and the response at 'c' due to an excitation at 'b' must include the effects of the reaction force such that

$$x_c = H_{ca}F_a + H_{cb}F_b \tag{41}$$

With some simple substitutions using the equations above, one can see that the frequency response function between point 'c' and 'b' with the tip of the cantilever beam restrained can be obtained from the frequency response measurements of the unconstrained original system as

$$\tilde{H}_{cb} = \frac{x_c}{F_b} = H_{cb} - H_{ca}H_{aa}^{-1}H_{ab} \tag{42}$$

In its simplest form, this is the basic equation for impedance modeling. This technique has been extended to handle multiple modifications simultaneously and used as a system modeling technique. (This impedance modeling approach will be used in the example section to illustrate truncation effects with the SDM approach.)

Structural Dynamic Modification Utilization Issues

The theory was presented that is applicable to all structural modification techniques. There are many issues pertaining to the success of the modification process, with the major items discussed in this section. The major issues pertain to truncating modes that define the modal data base and the lack of rotational DOF. However, other issues of scaling mode shapes, rigid body modes, higher frequency modes, and complex vs. proportional modifications are also important considerations. These are discussed in the following sections.

Truncation of Modes. The modes that are used to describe the modal data base will inevitably suffer from modal truncation. This is a severe limitation that is encountered in any experimental modal test. Only the lower order modes of the system will be obtained from the test data. Even the finite element model will have a limited number of modes to describe the modal database. However, the finite element model can extract additional modes, if necessary, when performing structural dynamic modifications.

Actually, the higher frequency modes are not the only modes of concern. Many times an experimental test is conducted to obtain the lower order flexible modes of the structure in a free-free configuration. All too often the rigid body modes are not extracted from the test data. Many times this is due to the poor quality of the data collected at such a low frequency with the instrumentation used to address the flexible modes of the system. In addition, many test engineers do not necessarily associate the rigid body modes as part of the description of the modal database, believing that only the flexible modes are of concern. The rigid body modes are a critical set of modes for structural dynamic modifications. Approximations have been attempted to estimate these necessary rigid body modes from a variety of different measurement and analytical techniques. Rigid body modes are discussed in a separate section.

Efforts have also been expended to estimate residual compensation terms to account for the higher frequency terms that have been excluded from the modal database due to truncation. Static correction terms, residual shapes and other approximations have been used to assist in the modification process to improve results. Many of these generally improve the modification results but often are limited because most structural systems require DOF that relate physical rotational information in almost any real structural change that is investigated. Rotational effects are covered in the next section.

Modal truncation is likely the most critical of all the modification errors that are encountered. Reference [26] contains several models to illustrate some of these truncation effects, while some of the more important models are presented in the Appendices. Appendix A presents a simple example to illustrate the effects of truncation for a simple 6 DOF system. Appendix B presents a modification of a free-free beam to approximate a simple support beam and a cantilever beam using the SDM process to illustrate some additional truncation effects. A system model is presented in Appendix C to show how system modes may be affected by truncation.

Rotational DOF. The experimental modal database is largely comprised of only translational DOF. The measurements obtained are almost exclusively translational DOF. The rotational measuring device has not found its place into affordable, economical, accurate use for experimental modal testing. Twenty years ago this was a major obstacle in all structural dynamic modification and system modeling studies, and it still plagues the analysis process today.

At that time, many different approaches were investigated to estimate rotational DOF. Approaches that use spline fits of measured data, analytical representations of curvature, and analytical expansion methods were numerical approaches to estimate these necessary DOF.

From the experimental side of the problem, it is very expensive to measure rotational DOF for most routine modal tests, which has led to many attempts to estimate rotational DOF. This began with closely-spaced accelerometers being added to the structure; later a housing was designed that incorporates two accelerometers to estimate rotational DOF. These approaches provide some information but suffer from measurement inaccuracies such as cross axis sensitivity. Others have attempted to intentionally mass-load the test structure with ‘calibrated’ mass adjustments to produce known effects, allowing for the ultimate estimation of rotational DOF. Further research and effort continues today to attain an accurate, economical device for measuring rotational DOF.

In order to illustrate the rotational DOF truncation effect, the cantilever modification of Appendix B is extended for rotational DOF truncation. The modal database contains different sets of translational DOF and rotational DOF to show the effects of truncation, shown in Appendix D. To further illustrate some additional points, the truncation of rotational DOF on the synthesis of FRFs is illustrated with discussion on the effect of modification of the cantilever using either the modal based approach or the impedance based approach. This model is discussed in Reference [29] and some of the pertinent results are shown in Appendix E.

Mode Shape Scaling. The experimental modal database must be obtained from accurate calibrated measurements. Mode shape scaling must be clearly identified and considered prior to performing any structural dynamic modification. Since the modification equations use the physical properties of mass, damping and stiffness in the modal equations of motion, the equations must utilize consistent units. For most experimental modal test results, the animation is simply a display of the ‘amplified’ shape characteristics and does not necessarily have any physical units. However, when the mode shapes are scaled in a particular fashion, such as unit modal mass, then there is a direct relationship between the mode shapes and physical quantities.

If improper calibration units are used for the collection of experimental data, the mode shapes will be scaled with a bias error. This will have a direct impact on the modal change matrix in modal space (the projection of the physical change in the system using the mode shapes of the system). Obviously, if the mode shapes are scaled incorrectly, then the projected change to the system in modal space will be directly affected.

Drive Point Measurements. During the collection of modal data, the drive point frequency response function is necessary in order to provide the proper calibration and scaling factors for the measured modes. The residues obtained from the modal parameter estimation process are sufficient for the definition of the system characteristics. However, the mode shapes form the frequency response function from information related to both the input and response mode shapes for each of the modes of the systems. Only the drive point measurement contains information that allows the residue to be related to the mode shapes.

This important point is the main reason why operating data cannot be used to determine the effects of structural dynamic modifications. Operating data does not have any scale information relative to the mode shapes of the system. While operating data is extremely useful in studying structural dynamic systems, this data contains no scale information. Drive point measurements are required in order to obtain accurately scaled mode shapes.

Rigid Body Modes. For many structural modification studies involving either the constraint of a structure to ground or the connection of one substructure to another substructure, rigid body modes are required. These modes are necessary in order to define the inertial characteristics of the component. If these rigid body modes are not included in the modal database for these types of structural changes, then significant errors will result. The rigid body modes are a required portion of the adequate description of any free-free modal component.

In many cases, the rigid body modes comprise a necessary part of the modal description of the system and cannot be ignored. If these modes are not included, then for all practical purposes the modal data base is truncated (even though the rigid body modes are the lowest modes of the database). Care must be exercised when incorporating rigid modes into the measured experimental flexible modes. These rigid body modes must be scaled in a manner that is consistent with the scaling of the measured flexible modes.

In addition, the rigid body modes obtained from a finite element model will have a frequency of 0 Hz and the modes will be repeated. This is generally not true of the rigid body modes obtained from testing. The tested rigid body modes will not have a frequency of 0 Hz and will most likely not be repeated. This is an important point when performing structural dynamic modifications. First, the complete mode set used to describe the system (rigid body modes and flexible modes) must be scaled in a consistent fashion. Second, many commercially available implementations do not properly handle repeated roots and often do not handle modes that have a frequency at 0 Hz. Since most tested structures do not have 0 Hz rigid body modes, this is typically not a problem. However, caution needs to be exercised when augmenting a modal data base with analytically derived rigid body modes or when implementing a structural dynamic modification procedure using a free-free component described by the finite element modes of the structure. Care must be taken to assure that the software can adequately handle both 0 Hz rigid body modes as well as repeated roots.

Complex vs. Proportional Modes. Many times, proportional modes are used for the approximation of structural dynamic systems. Typically, this is a good approximation for most structural dynamic studies. However, when dealing with the structural dynamic modification process, a complex mode solution is required to obtain accurate modified system characteristics. Reference [8] contains several models that illustrate some of the effects of using proportional mode approximations when performing modifications that truly require a complex mode solution scheme. The results of those studies can be summarized with the following statements regarding proportional mode and complex mode solutions.

When in doubt, a complex mode solution will always produce the proper mathematical solution. However, a proportional mode solution generally will produce a reasonably good solution when the starting modes are indeed proportional and the changes involve only mass and stiffness. These types of changes on a proportional mode approximation generally do not cause the solution to become very complex – the modes tend to remain close to proportional. However, a proportional mode solution will often contain errors when damping modifications are investigated. As each individual modification is investigated, errors will accumulate because the proportional approximation ignores the complex phasing that results from the modified system. In this case, a complex mode solution scheme is necessary.

When a proportional mode solution is used and the initial unmodified modes are not real normal modes or proportional modes, but actually contain some phasing in the description of the modal database, then any changes of mass, damping or stiffness will always tend to accumulate error as modifications are performed. In this case, when the starting modes are not real normal modes, then one of two approaches are necessary. One approach is to employ the full state space solution scheme and utilize the complex mode structural modification procedure. This will be necessary – especially if the unmodified modes are believed to be truly complex. However, if the modes are believed to be proportional but the measurements or modal parameter estimation process caused the modes to have phasing that is believed to not truly exist, then the mode shapes should be rescaled to obtain a real normal mode approximation of the shapes. This second approach is needed if the modes are actually believed to be more appropriately real normal modes. This will tend to minimize errors from the modification process.

Other Issues. Of course, there are a variety of other issues that cause problems for the structural dynamic modification process. These relate to simple and often obvious matters that are overlooked, such as geometry definition, accurate calibration, consistency of units (weight vs. mass descriptions and feet vs. inches), lack of measured DOF for connection of structural modification elements, and a myriad of user blunders that cause errors in the modification procedure. Careful execution of this approach will yield accurate results (excluding the effects of truncation and rotational DOF that have been discussed).

Summary

Structural dynamic modification has been available for over two decades. The technique is useful for the study of structural changes to a system to determine optimum design changes. Issues of truncation and lack of rotational DOF have caused difficulties with the technique since its inception. Some of these problems were discussed along with some of the other effects that need to be considered.

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Appendix A: 6 DOF Example

A simple 6 DOF model is used to illustrate some key points pertaining to truncation and the matrices involved in the SDM process.[26] The model is shown in Figure A1. Note that the physical set of equations involve matrices that have coupling of the various DOF of the model. The eigensolution uncouples those equations to create a set of linearly independent SDOF systems, illustrated in Figure A1. Notice that all the modes are not retained in the description of the system (as shown by several modes in the cross-hatched area). These modes are not necessarily needed in order to obtain an acceptable solution regarding the system response.

To illustrate some of the effects of the structural dynamic modification process, a simple structural change is shown in

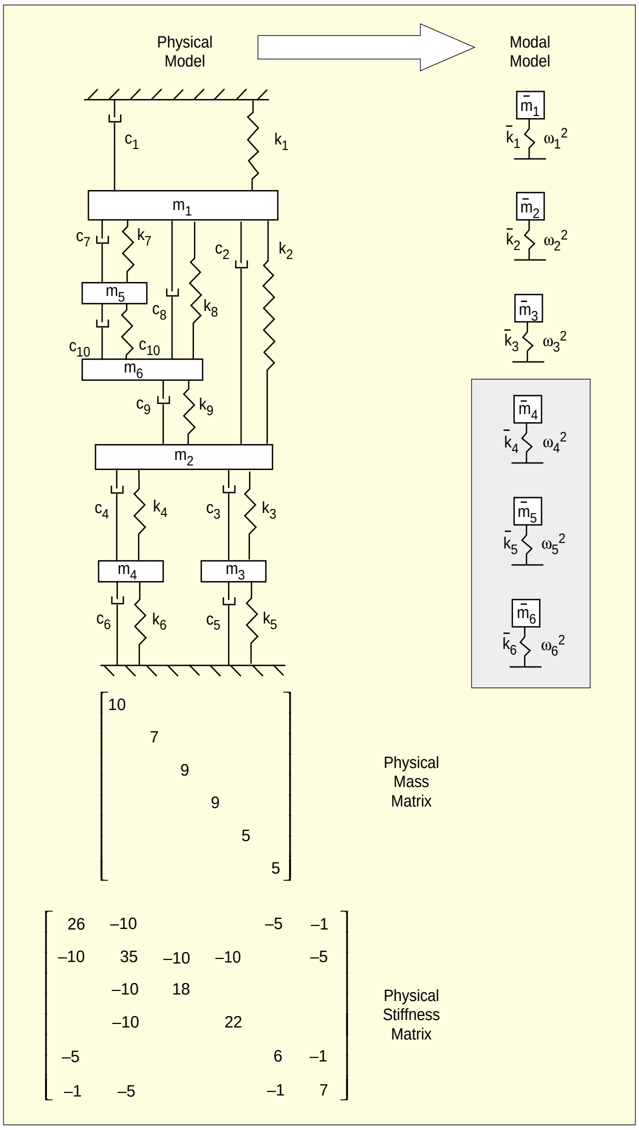


Figure A1. 6 DOF physical and modal model.

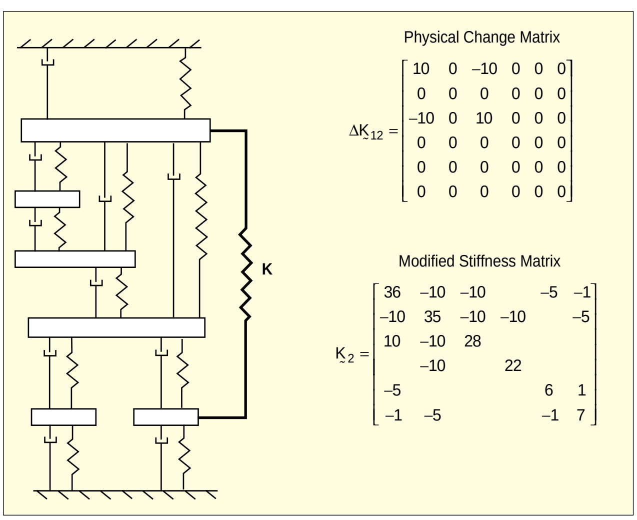


Figure A2. 6 DOF model with physical modification.

Figure A2. This change appears very discrete in the matrices of the physical model. However, when a structural modification is performed, the projection of the physical change matrix to modal space produces a very significant amount of coupling to the original unmodified modes of the system, shown in Figure A3. While the original modal stiffness is clearly a diagonal representation in modal space, the change in modal

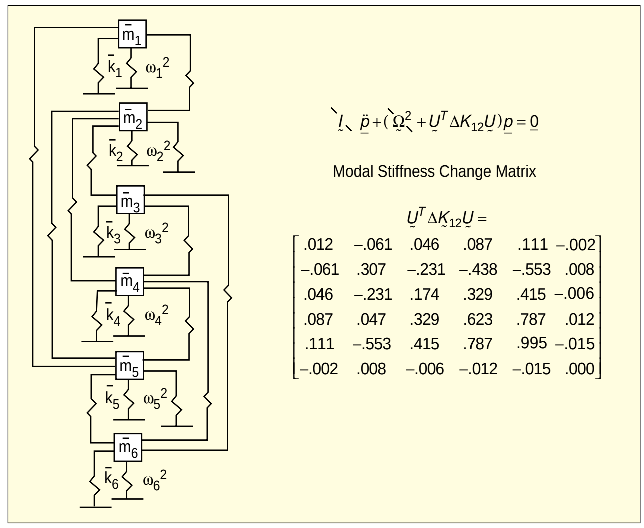


Figure A3. 6 DOF model with modal change matrix.

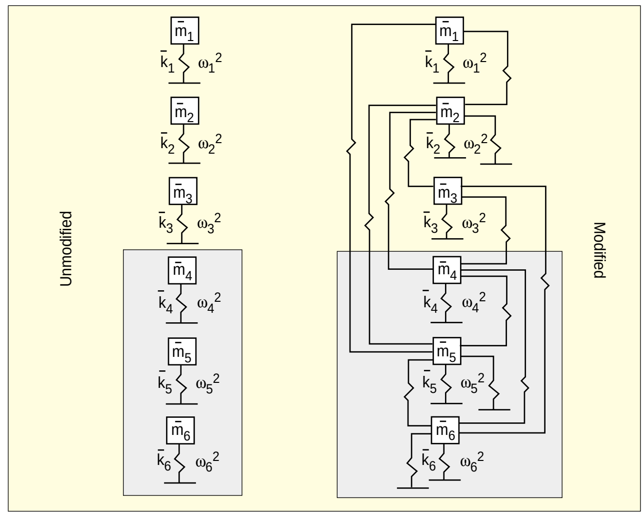


Figure A4. 6 DOF model truncation for an unmodified and a modified system.

stiffness is far from diagonal. The effect of the physical modification projected in modal space is to couple every one of the unmodified modal SDOF systems to each other and to reconnect them all with additional stiffness back to ground.

With an understanding of this effect, the modal truncation problem can be clearly seen. For the unmodified system, the truncation effect is related to the inclusion or removal of modes to describe the database. However, for the modified system, there is an effect of the projected change in modal stiffness due to physical modifications. The effects of truncation can be conceptually stated as follows: If the change in stiffness that crosses the imaginary boundary between the kept modes and the unkept modes is very small, then the effects of truncation are minimal. But if the change in stiffness between the kept modes and unkept modes is very large, then the effects of truncation are significant. Unfortunately, the unkept modes are not available to ascertain their effects.

Appendix B: Free-Free Beam for Simple Support and Cantilever Beam Modifications

A simple free-free beam is used to describe a modal database with 5 modes (2 rigid body and 3 flexible) for a structural dynamic modification involving only two stiffness changes: one to simulate a simple support and one to simulate a cantilever.[26] The various beam configurations are shown in Figure B1 and the results of the structural modification (which involves only two stiffness modifications) are shown in Table B1. The frequencies from the exact solution are shown for reference and

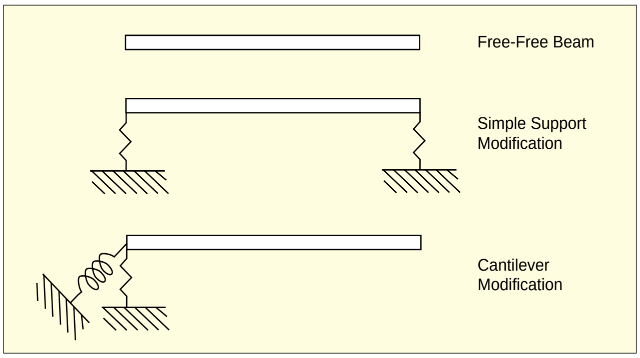


Figure B1. Modal beam and proposed modifications.

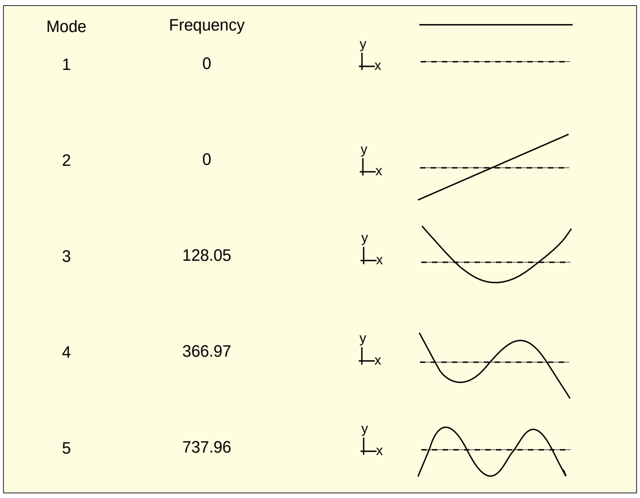


Figure B2. Free-free beam modal data.

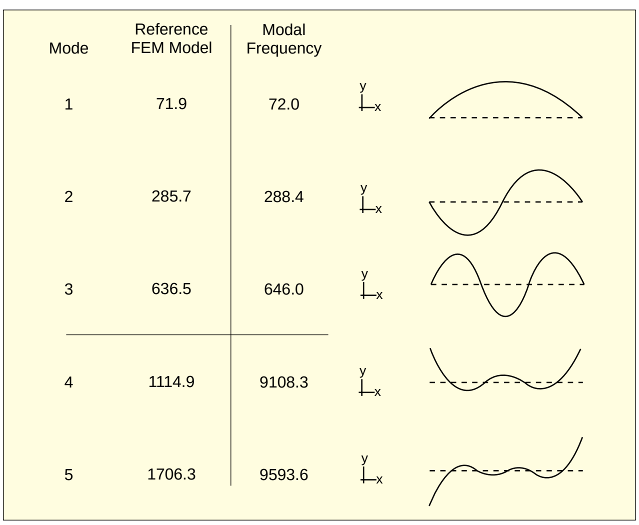


Figure B3. Simple support beam modal data.

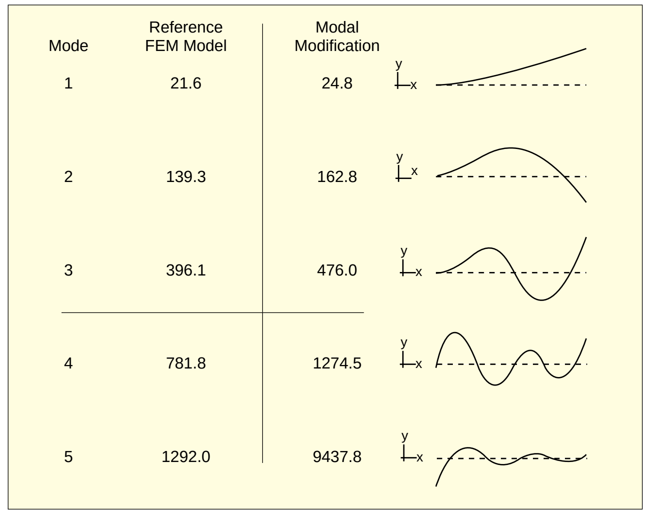


Figure B4. Cantilever beam modal data.

$$U_{12} = \begin{bmatrix} 0.8282 & 0.0043 & 0.2687 & 0.0053 & 0.4918 \\ 0.0058 & 0.6187 & 0.0019 & 0.7856 & 0.0032 \\ 0.5604 & 0.0000 & 0.4094 & 0.0003 & 0.7200 \\ 0.0000 & 0.7856 & 0.0000 & 0.6187 & 0.0002 \\ 0.0079 & 0.0000 & 0.8719 & 0.0001 & 0.4897 \end{bmatrix}$$

$$U_{12} = \begin{bmatrix} 0.7306 & 0.4026 & 0.2389 & 0.3488 & 0.3541 \\ 0.6659 & 0.2737 & 0.2239 & 0.3860 & 0.5315 \\ 0.1465 & 0.8298 & 0.0202 & 0.1360 & 0.5207 \\ 0.0335 & 0.2579 & 0.8307 & 0.2371 & 0.4314 \\ 0.0121 & 0.0891 & 0.4498 & 0.8091 & 0.3673 \end{bmatrix}$$

Figure B5. U_{12} projection matrix for the simple support beam and cantilever beam modifications.

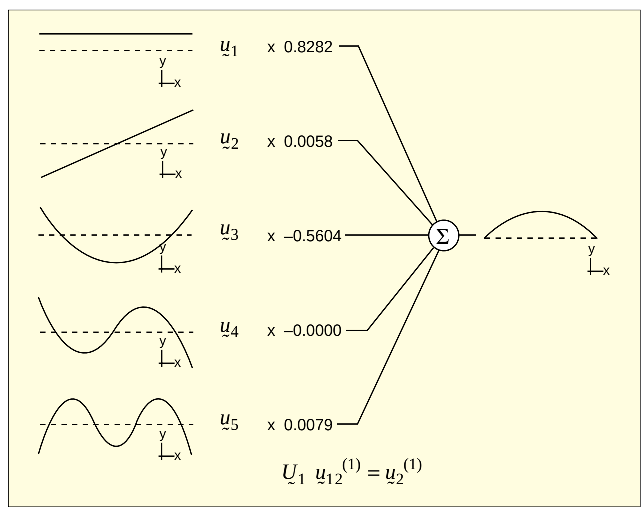


Figure B6. Free-free mode combinations for simple support beam modification for mode 1.

to assist in the evaluation of the accuracy of the structural modification solution. The original frequencies and mode shapes for the free-free beam are shown in Figure B2. The modified frequencies and mode shapes for the simple support modification study are shown in Figure B3. The modified frequencies and mode shapes for the cantilever modification study are shown in Figure B4.

Table B1. Frequencies of different modifications.

No.	Free	Simple Support Ref.	Simple Support SDM	Cantilever Ref.	Cantilever SDM
1	0	71.9	72.0	21.6	24.8
2	0	285.7	288.4	139.3	162.8
3	128	636.5	646.0	396.1	476.0
4	367	1114.9	*9108.3*	781.8	*1274.5*
5	738	1706.3	*9593.6*	1292.0	*9437.8*

Note: Frequencies in *italics* are approximations of constraint modes.

Before the lower frequencies are discussed, it is important to note that the last two frequencies of each set for the modified modes are very high in frequency. This is due to the fact that only 5 modes are used for the modification of the modal system. In both cases, these two higher frequency modes are approximations of the constraint modes of the system. These are approximations of the modes necessary to constrain the system to ground. Discussion of these constraint modes are beyond the scope of the information presented here but can be found as part of the description of the mode sets available for component mode synthesis.

For the simple support modification, the results compare very well to the exact solution. The results are very acceptable,

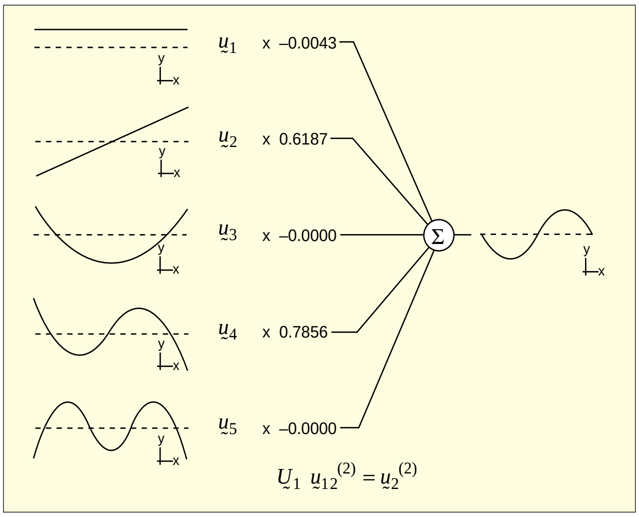


Figure B7. Free-free mode combinations for simple support beam modification for mode 2.

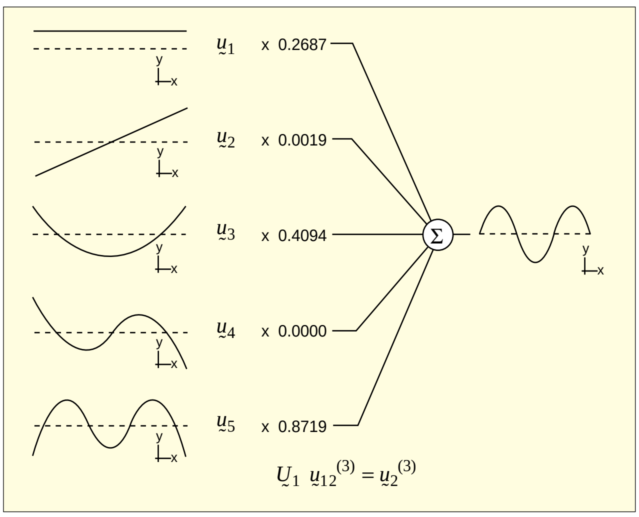


Figure B8. Free-free mode combinations for simple support beam modifications for mode 3.

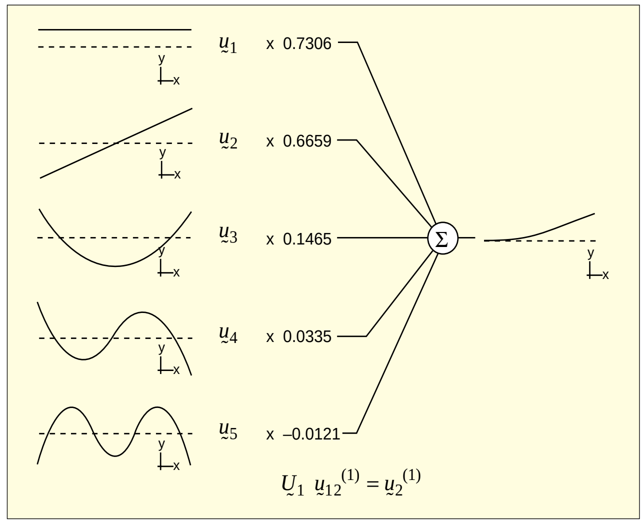


Figure B9. Free-free mode combinations for cantilever beam modification for mode 1.

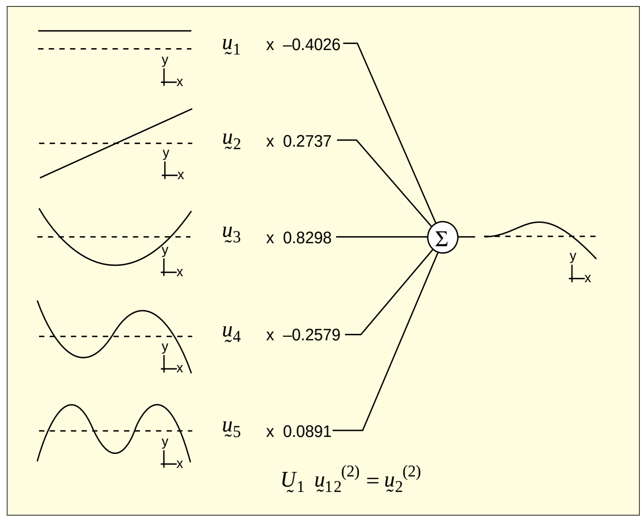


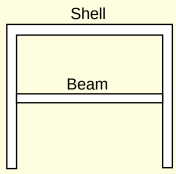
Figure B10. Free-free mode combinations for cantilever beam modification for mode 2.

especially considering that only 5 modal DOF were used for the modal modification process. However, the cantilever beam, which also has only two stiffness modifications to describe the change, has much more error associated with the predicted modified modes. The reason for this can be seen upon investigating the U_{12} matrix. This provides information regarding the linear combinations of the unmodified modes that form the final modified modes. The U_{12} matrix is shown in Figure B5 for the simple support and cantilever beams. Each column of this matrix identifies the proportions of each mode necessary to form each final system mode for both modification cases. It is this linear combination effect that can be graphically observed in the following figures for the simple support and cantilever beam modifications. The simple support beam is shown in Figures B6, B7 and B8 for modes 1, 2 and 3, respectively. The cantilever beam is shown in Figures B9 and B10 for modes 1 and 2, respectively. Visual observations of these figures along with the U_{12} matrices in Figure B5 reveal the reason for the errors.

Basically, the simple support modified modes can very easily be accomplished from the linear combinations of the modes from the free-free beam. That is to say, the starting modes and final modes are very similar. It is a very easy task to form linear combinations from the original unmodified modes to create the final modified modes. However, the same is not true for the cantilever beam modification. In fact, it is extremely difficult to create the final modified cantilever configuration from the starting modes of the original unmodified free-free modes; many more modes of the free-free beam are necessary in order to improve the final modified results. The final modes of the cantilever system do not appear similar to the starting modes of the unmodified free-free beam and many modes are necessary in order to improve the accuracy of this modification.

Appendix C: Free-Free Beam and Shell for Development of Frame System Model

Two simple modal components for a beam and shell are used to form a system model.[26] The intent of this model is to show that all of the final modified system modes are not equally affected by the modal truncation. The reference frequencies of the full model of the structure based on physical matrices are shown in Figure C1, along with the modes resulting from the modification to form the system from two modal components. (Note that the beam modal description is the same as that used in Appendix B.) In particular, the final system modes 4 and 5 will be discussed to illustrate some of the effects of truncation for different final modes of the system. The U_{12} matrix is shown for the first six modes in that figure; the 4th and 5th columns relate to the scaling coefficients for the final modified 4th and 5th system modes, respectively. The 4th mode of the modified system appears to be significantly affected by truncation whereas the 5th mode appears to have almost no error associated with the effects of truncation. A graphical presentation of the mode combination is shown in Figures C2 and C3 for the



Comparison of System Modes Resulting from a Modal Model Developed from 2 Modal Components

Reference Frequency	Modification (System Model)
32.4	36.5
108.0	109.0
115.0	119.0
128.0	162.0
305.0	305.0
350.0	373.0

Transformation for First 6 System Modes

$$\underset{\sim}{U}_{12} = \left[\begin{array}{cccccc}
.93 & 0 & -.28 & 0 & -.04 & .06\\
0 & .83 & 0 & .36 & 0 & 0\\
.01 & 0 & .68 & 0 & -.22 & .09\\
0 & .39 & 0 & .18 & 0 & 0\\
.01 & 0 & .04 & 0 & .87 & -.39\\
0 & -.04 & 0 & .10 & 0 & 0\\
-.01 & 0 & 0 & 0 & -.05 & -.30\\
0 & 0 & 0 & .02 & 0 & 0\\
0 & -.01 & 0 & -.01 & 0 & 0\\
0 & 0 & .01 & 0 & .01 & -.04\\
0 & 0 & .01 & 0 & .02 & .03\\
0 & 0 & 0 & .01 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
\hdashline
.36 & 0 & .68 & 0 & .18 & .02\\
.08 & -.02 & .09 & .04 & -.23 & -.55\\
0 & -.33 & 0 & .74 & .01 & .03\\
0 & -.23 & 0 & .52 & 0 & 0\\
-.03 & 0 & .01 & 0 & .32 & .66\\
0 & .02 & 0 & -.06 & 0 & 0\\
-.01 & 0 & 0 & 0 & 0 & -.05
\end{array}\right]
\begin{array}{l}\text{Shell}\\ \\ \text{Beam}\end{array}$$

Figure C1. Free-free mode combinations of beam and shell for frame system model development.

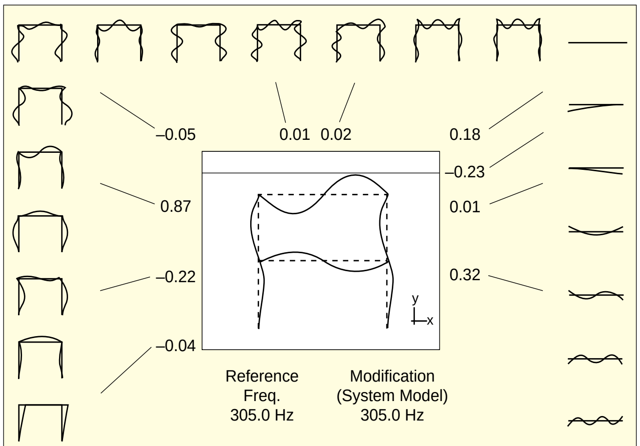


Figure C3. Free-free mode combinations of beam and shell for frame system model for system mode 4.

final 4th and 5th system modes, respectively.

Upon reviewing the 5th column of the U_{12} matrix, the original unmodified modes of the two separate components are sufficient in linear combinations to form the final 5th system mode of the structure. In fact, the 2nd free-free flexible mode of the beam and the 5th mode of the shell appear to be the most dominant modes from a visual perspective, in the final 5th mode of the system; this is confirmed upon reviewing the U_{12} matrix. These two modes naturally merge to form the final 5th system mode characteristic; that is why this mode does not suffer because of truncation.

The same is not true of the 4th modified system mode. This system mode requires many more modes of the free-free beam. In the final 4th system mode characteristic, the mode shape related to the beam portion of the system appears to behave very much like a beam with both ends fully built-in. Recalling the cantilever beam of Appendix B, the cantilever modification required many modes for an accurate approximation. The same is true of the beam portion of this frame system model – the

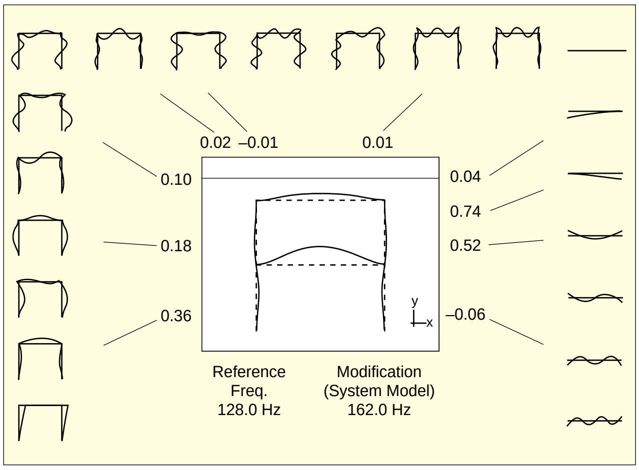


Figure C2. Free-free mode combinations of beam and shell for frame system model for system mode 4.

Figure D1. Cantilever beam used to form modified pinned configuration.

beam suffers from modal truncation and more modes are required to accurately model this particular system mode.

It is very important to notice that truncation does not have the same effect on all the resulting modified system modes. The 4th modified system mode (128 Hz) is seriously affected by modal truncation whereas the 5th modified system mode (305 Hz) is relatively unaffected by truncation.

Appendix D: Cantilever Modification to Illustrate RDOF Truncation

The effects of truncation are critical to any structural modification study. In many modifications, the rotational DOF are much more critical than the translational DOF. An extension of the cantilever modification in Appendix B is used to illustrate this effect. The modification of the cantilever is made with 5 free-free modes and 10 free-free modes. However, the modal database is evaluated with different sets of modes: First with only 5 translational and 5 rotational DOF, followed by 10 translational and 5 rotational DOF, and then finally with 10 translational and 10 rotational DOF. The model is shown in Figure D1 and the results are shown in Table D1.

The inclusion of additional translational DOF in the intermediate case (10 TDOF and 5 RDOF) has minimal effect on the improvement of the modification prediction. Only when extra rotational DOF are included does the modification prediction improve. This clearly shows that these rotational DOF are far more critical for this modification. Since many real world modifications of structural elements involve rotational effects, the lack of rotational DOF will generally prove to be responsible for the majority of the truncation effects.

Appendix E: Pin Modification at the Tip of a Cantilever Beam Using FRF Modification

To further illustrate the effects of truncation, a simple, FRF-based modal modification prediction is discussed.[29] The

Table D1: Frequencies of different truncated set combinations

No.	Ref. Freq. (Hz)	5 Modes (Hz) (1-5 TDOF) (1-5 RDOF)	10-5 Modes (Hz) (1-10 TDOF) (1-5 RDOF)	10 Modes (Hz) (1-10 TDOF) (1-10 TDOF)
1	21.6	24.8	24.8	22.2
2	139.6	162.8	162.6	144.9
3	398.6	476.0	473.7	411.4

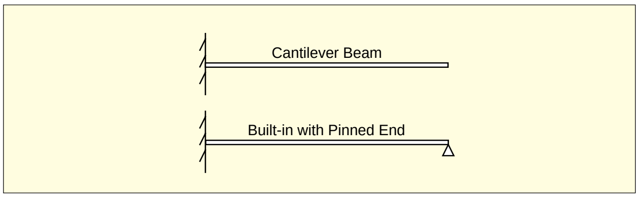


Figure E1. Cantilever beam and pin modification.

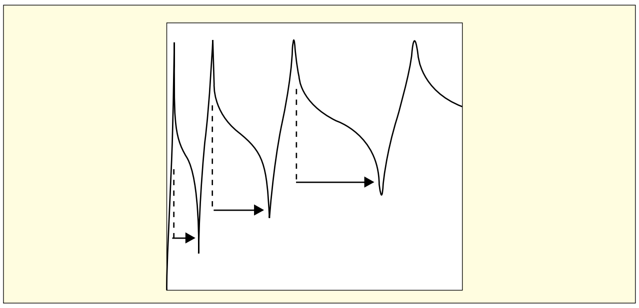


Figure E2. Cantilever beam drive point FRF for pin modification at tip of the cantilever beam.

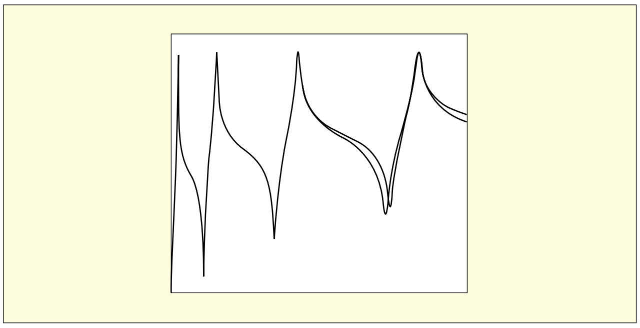


Figure E3. Cantilever beam synthesized FRF for all modes, 7 modes and 5 modes for translational effects at the tip of the beam.

simple cantilever in Appendix B is used again to study the effects of a pin modification at the free end of the cantilever beam, shown in Figure E1. In order to understand the modification and the results anticipated, a drive point measurement at the tip of the cantilever (where the pin connection modification is to be made) is shown in Figure E2. As the stiffness of the spring used to approximate the pin modification is increased, the frequencies of the modified cantilever beam will increase. However, the modified frequencies of the cantilever beam with the pin will never become larger than antiresonant frequencies of the unmodified cantilever viewed from the point of constraint. These antiresonances are physical ‘roadblocks’ to the shift of the natural frequencies of the modified system. Actually, the antiresonances are the frequencies where the tip of the beam has no motion. The physical interpretation is that the cantilever beam appears to have no motion of the tip at the frequencies that are analogous to adding a pin at this location. Therefore, the antiresonances specify the shift in frequency when the stiffness of the constraint approaches infinity.

The synthesis of a frequency response function can be performed using the modes of the system with either residues or mode shapes for the formulation of the function. The poles or modes are global parameters of the system. As higher modes are truncated, the peaks associated with the lower order modes are relatively unaffected in terms of their frequency location. However, the same is not true for the antiresonances, which are strongly affected by truncation. As more and more modes are truncated, the antiresonances shift upwards in frequency (see Figure E3).

This further explains the modal modification truncation effect. As more and more modes are truncated from the modal database, the antiresonances shift higher in frequency. Since the antiresonances are ‘roadblocks’ to the modification, the

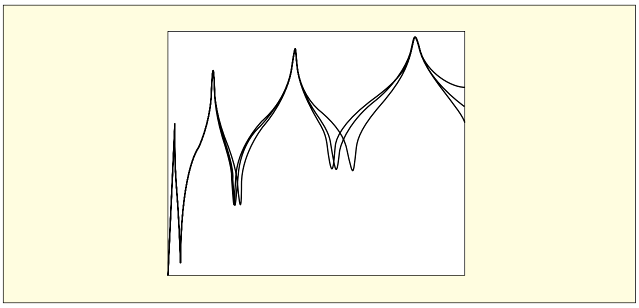


Figure E4. Cantilever beam synthesized FRF for all modes, 7 modes and 5 modes for rotational effects at the tip of the beam.

upward shift in frequency helps explain why the modal modification always produces modified frequencies that are higher than the true modification. The truncation effect is very clearly seen in the modal database but it is now apparent that the FRF modification approach will also suffer from truncation when the FRFs are obtained from a synthesized function.

To further illustrate the truncation effect, a rotational DOF at the tip of the beam is used to synthesize a FRF as seen in Figure E4. As the modes are truncated from the database, the antiresonances shift upwards in frequency. However, they are observed to shift substantially more than the translational FRFs seen in Figure E3. Therefore, the effects of truncation will be much more pronounced on the rotational DOF when compared to the translational DOF. This is the main reason why the modifications of the cantilever suffer much more from truncation when compared to the simple support modification of Appendix B.