Effects of Windowing on the Spectral Content of a Signal

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Fourier analysis is commonly used to estimate the spectral content of a measured signal. When choosing the appropriate window, one needs to be aware of its advantages and pitfalls in order to fit the measurement situation. The following deals with some practical considerations on the effects of windowing.[1]

The Fourier series assumes periodicity of the signal in the time domain. An FFT is actually a Fourier Series performed upon an interval, $T_{span} = n\Delta t$ where n is the number of samples observed and Δt is the constant time between samples. Since the sampled time signal may not exactly contain an integer number of periods, this assumption may not be truly satisfied.

In effect, the truncation of the original signal corresponds to its multiplication with a Rectangular window of length T_{span}. The Fourier series then assumes that the signal is the succession of versions of this truncated signal in the time domain leading to a spectrum with harmonic components at frequencies equal to multiples of $\Delta f = 1/T_{Span}$.

Let us examine the situation with a sine wave of frequency f_0. In theory the corresponding spectrum is a peak at f_0. When a noninteger number of periods is acquired, this results in signal leakage, characterized by the smearing of the spectrum. Figure 1 illustrates this phenomenon by comparing three cases, where f is the sine wave frequency and Δf the frequency resolution.

Case 1(a) allows us to determine the ideal situation where an integer number of periods (200) is set for the signal generator. The corresponding frequency is f_0 = 508.626 Hz. In practice this case is not likely to occur, because the frequency that is being measured rarely falls on a frequency line. On the other hand, case 1(b) represents a typical situation where the leakage is clearly visible. Here f has been slightly decreased, which results in a non-integer number of periods within T_{span}. The maximum leakage is obtained in case 1(c). Why? The answer lies in the spectrum of the window as shown in Figure 2(a).

The FFT emulates a bank of parallel bandpass filters with the center frequencies exactly centered on integer multiples of Δf. The width and shape of each filter is identical and are given by the spectrum of the observation window shown in Figure 2. Note that the filter shape is characterized by multiple lobes separated by zero values at multiples of Δf and that all filters in the bank 'overlap.'

When f of an applied sine corresponds exactly to a filter center-frequency [case 1(a)], only that filter will respond because f corresponds to an amplitude notch of all other filters in the bank. Conversely, if f is not exactly on a frequency line [case 1(b)], the energy at f is smeared over adjacent frequencies because the secondary lobes of all other filters overlap f with nonzero gain and these filters respond in proportion to this gain. This perverse effect is maximum when $f = f_0 - \Delta f/2$ [case 1(c)], since the frequency f coincides with the peak of each side lobe.

If side lobes could be reduced in amplitude, this error would decrease as well. This is why people have used a number of windows to weight the truncated signal such that the starting value and the ending value are zero. This produces a signal that appears periodic in T_{span}, meeting the basic assumption of the Fourier Series. Weighting avoids the sharp discontinuities induced by the Rectangular window and yields reduced-amplitude side lobes as desired. Figure 2(b) shows the spectrum of the well known Hanning[2] window. Observe that the amplitude of the first side-lobe is reduced from –13.2 dB to –32.2 dB. More importantly, notice that the amplitudes of subsequent side-lobes fall off at 60 dB/decade as opposed to 20 dB/decade for the Rectangular window.

These improvements come at a cost. The width of the primary lobe essentially doubles, eliminating the first set of zero-amplitude points. The primary lobe of the Rectangular window has a –3 dB bandwidth of 0.85 Δf. That of the Hanning window is increased to 1.4 Δf. However, the benefits far outweigh the cost as shown in Figure 3. Here the Hanning window is applied to the three cases previously examined. Indeed results with the Hanning window are close to case 1(a) done with the Rectangular window (ideal case).

However, the Hanning window exhibits a deficiency. Like the Rectangular window, its primary lobe has significant curvature or 'ripple' across the $\pm\Delta f$ band. When a sine falls "exactly between cells," its amplitude is reported 15% (–1.42 dB) lower than it would be at the filter center-frequency. The Rectangular window exhibits this same fault, but more pronounced at 36% (–3.92 dB).

When the application requires an accurate measure of peak amplitude (e.g. rotating machinery), the Flat-Top window is usually selected. Its spectrum is characterized by a nearly flat main lobe across $f_i \pm\Delta f$, which reduces maximum amplitude error to 0.1%! As for the side lobes, their amplitudes remain at –70 dB below that of the main lobe, which strongly reduces leakage. However this window must be used with care, particularly if the periodic signal of interest is 'buried' in broadband noise. The Flat-Top window should only be applied to clean periodic waveforms. It is indeed a poor choice for random-signal or mixed-signal analysis because it lacks selectivity. A Hanning

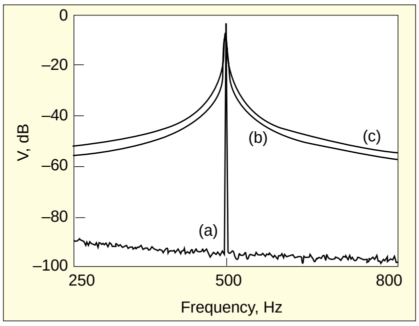

Figure 1. Spectrum of a sine wave with the Rectangular window: (a) $f = f_0$; (b) $f = f_0 - \Delta f/8$; (c) $f = f_0 - \Delta f/2$.

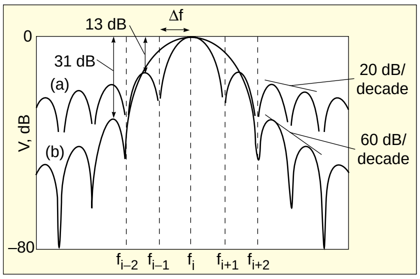

Figure 2. Comparison of spectra for the Rectangular window (a) and the Hanning window (b).

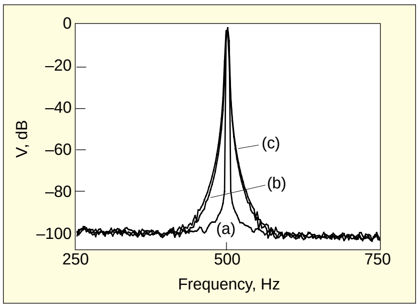

Figure 3. Spectrum of a sine wave with the Hanning window: (a) $f = f_0$; (b) $f = f_0 - \Delta f/8$; (c) $f = f_0 - \Delta f/2$.

Table 1. Comparison of windows for FFT analysis.

Window	Advantages	Pitfalls	N_b	Applications
Rectangular	Raw data Identification of closely-spaced frequencies	Leakage	1 Δf	Transient (Impact testing)
Hanning	Little leakage Frequency accuracy	Some amplitude error	1.50 Δf	Periodic, random
Flat-Top	Little leakage Amplitude accuracy	Large noise bandwidth No roll-off	3.43 Δf	Periodic
Exponential	Shorter T_{span} (quick measurements for modal analysis)	Adds artificial damping	Variable	Impact testing (Lightly damped structures)

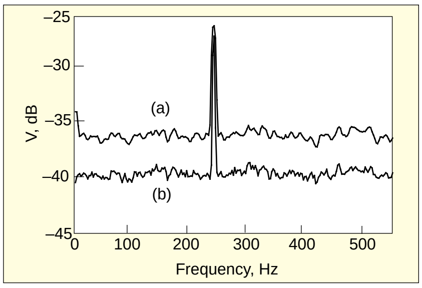

Figure 4. Comparison of spectra of a sine wave buried in random noise with: (a) the Flat-Top window; and (b) the Hanning window.

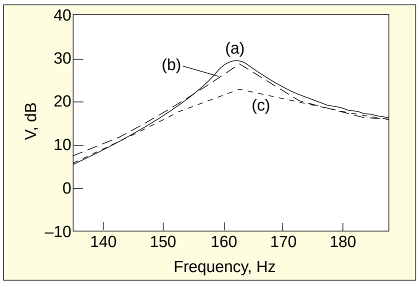

Figure 5. Transfer functions: (a) baseline; (b) with Rectangular window; (c) with Exponential window.

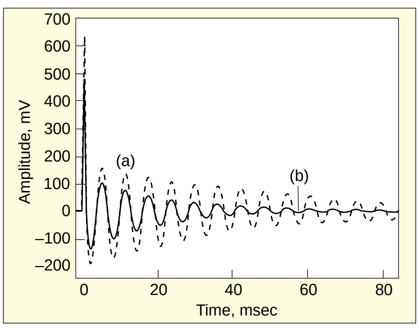

Figure 6. Time responses: (a) with Rectangular window; (b) with Exponential window.

window is a far better choice when trying to find a tone masked by random background. The Flat-Top window lacks selectivity for two reasons. First, its primary lobe is over twice as wide as that of the Hanning window suppressing additional zeros. Second, there is no roll-off of the side-lobes with frequency. Both of these characteristics render the Flat-top window sensitive to broadband noise, compromising its ability to 'find' a masked tone.

A window's sensitivity to broadband random noise is standardly characterized by its equivalent noise bandwidth N_b. This one-number description of a complicated shape may be found as follows. Consider an ideal (unity gain) rectangular 'brickwall' bandpass filter and an FFT filter that results from applying a window in time domain. When both filters are excited with the same random signal, N_b would simply be the frequency width at which the 'brickwall' filter passes the same power as the FFT filter. N_b is the 'Hz' reported in a g^2/Hz Power Spectral Density measurement.

While a Flat-Top window has a noise bandwidth of 3.43 Δf, a Hanning window exhibits an N_b of only 1.5 Δf. Note that a Rectangular window has a noise bandwidth equal to Δf, but it is unsuitable for random analysis due to its previously described leakage. Figure 4 compares spectra of a sine wave in random noise using the Hanning and Flat-Top windows. Note the superior dynamic range between the sine peak and the noise floor provided by the Hanning window.

So when would the Rectangular window be actually an appropriate choice? When the waveform is a transient and the window is large enough to contain the entire transient. This would apply to impact testing for instance, where the response decays from the time of impact. Note that T_{span} should be chosen large enough such that the response has sufficiently decayed to avoid any leakage due to the discontinuity at the block end.

When T_{span} is not large enough to contain the entire transient (as with lightly damped structures), leakage errors can result from the truncation of the signal. Often people use the Exponential window to ensure a sufficient decay at $t = T_{span}$. This approach is not always adequate, because it adds some artificial damping to the transfer function, as shown in Figure 5(c). Here a Single Degree of Freedom module from the Mentor[3] system provides the desired transfer function for this example. The damping coefficient is adjusted to obtain a lightly damped system. Then the averaged transfer function is measured by 5 impact tests, with excellent frequency resolution (large T_{span}). This constitutes our baseline [Figure 5(a)]. To exaggerate the different windowing effects in this example, the sampling parameters are modified such that T_{span} = 98.304 msec. The natural frequency of the baseline is set to ω = 162.125 Hz in order to be close to a frequency line. Hence the leakage observed in case 1(a) is avoided. Then the impact test is performed with the Rectangular window [Figure 5(b)] and the Exponential window chosen with an appropriate decay rate [Figure 5(c)]. Although both windows give a good estimate of ω, the Rectangular window achieves a better estimate of the amplitude (error of 1 dB as opposed to 6 dB with the Exponential window). The time responses are displayed in Figure 6. Table 1 summarizes the different characteristics of the windows mentioned here.

References

1. *Discrete-Time Signal Processing*, A. V. Oppenheim and R. W. Schafer, 1989.
2. Named after the Austrian meteorologist Julius Von Hann. It is in fact a corruption of the name Hann Window, perhaps through association with the Hamming Window named after the mathematitian R. W. Hamming.
3. The *Mentor* is a self-contained integrated signal analysis training system, G. F. Lang, Data Physics Corporation, 2001.

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