Generating Non-Gaussian Vibration for Testing Purposes

David O. Smallwood, Albuquerque, New Mexico

There is increasing interest in simulating vibration environments that are non-Gaussian, particularly in the transportation arena. A method is presented here for generating time histories that are realizations of a zero mean, non-Gaussian random process with a specified spectrum, skewness, and kurtosis. A zero-memory nonlinear (ZMNL) monotonic function $y = f(x)$ is generated to convert a zero mean Gaussian realization x with a specified autospectral density (ASD) into a non-Gaussian waveform y. The transformation is generated using the density method. The zero crossings are preserved in the transformation, which preserves most of the spectral information. The nonlinear transformation does introduce some harmonic distortion of the spectrum that can be reduced in an iterative process. The method does not preserve the phase structure between frequencies and is unable to match higher-order spectra. The resulting time history can then be used in a simulation or reproduced on a shaker using commercially available waveform reproduction software.

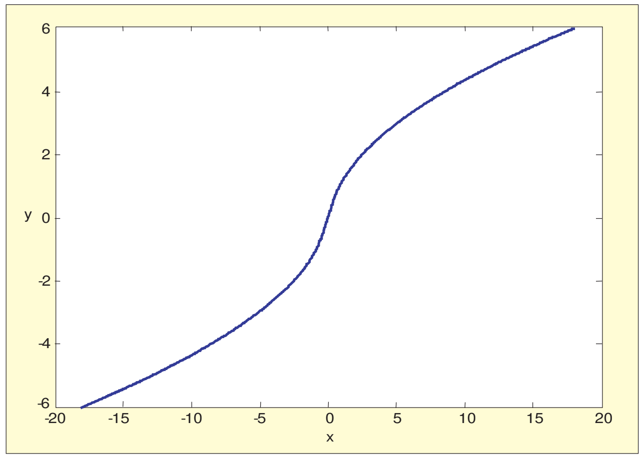

Figure 1. ZMNL function, S = 0, K = 2.

The generation of non-Gaussian noise has had a renewed interest in the defense industry for two reasons. The first is the realization that many surface transportation and wave environments are non-Gaussian. The second is the development of shaker control systems that could replicate long time histories that are non-Gaussian. The original, and many current shaker control systems, for generating random vibration tests generate only Gaussian random noise. However, waveform replication techniques now allow the reproduction of any waveform whose characteristics are within the bounds of a shaker.

Many articles have been written on the subject of generating non-Gaussian waveforms. This article uses the method of a zero-memory nonlinear (ZMNL) function. This method was proposed as far back as 1967.[1,2] At that time, analog circuits were used to implement the methods. The basic method relies on the equation:[3]

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| \tag{1}$$

where:
$f_Y(y)$ = probability distribution of the random variable Y
$f_x(x)$ = probability distribution of the random variable X

It is understood that in Equation 1, x stands for $x = g^{-1}(y)$, where $y = g(x)$ and the density function depends on y, not x. If $y = g(x)$ is not monotonically increasing, the function must be broken into parts, and each part handled separately. For the purposes of this article $y = g(x)$ will be restricted to a monotonically increasing function. Therefore, dx/dy is always positive. For this article, the distribution of X will be assumed to be a zero mean, unit-variance, Gaussian distribution. If the distribution $f_Y(y)$ is known (the distribution of $f_X(x)$ is assumed to be Gaussian), dx/dy can be found, and integration will yield $x = g^{-1}(y)$. Inversion will yield $y = g(x)$. $y = g(x)$ is the ZMNL function. The ZMNL function will be scaled so that a unity variance X will yield a unity variance Y. In many real applications the complete distribution $f_Y(y)$ is not known, but estimates of the skewness and kurtosis are known. The idea is to find a relatively simple function $y = g(x)$ so that the distribution $f_Y(y)$ is zero mean unit variance, with a specified skewness and kurtosis. The conversion of a realization of x to y using a ZMNL function will always produce an autospectral density for y that is different than the autospectral density of x.[4] The effect is to add harmonics to all the Fourier components of x. This will make the transformed data appear 'whiter' than the original data. However, most of the spectral information is contained in the zero crossings,[5] which are preserved, and if the nonlinearity is not too great, the spectral change is usually acceptable and the autospectral density of Y will be near the autospectral density of X.

Based on a paper presented at the 74th Shock & Vibration Symposium, Virginia Beach, VA, October 2004.

Implementation

In this implementation, x a realization of X, is generated with the desired autospectral density shape and scaled to a unity variance. The realization is converted to a realization of Y using $y = g(x)$. The realization y is then rescaled to the desired variance and used in a computer simulation or reproduced on a shaker using waveform replication techniques if a vibration test is desired. Five constraints have to be met – the area under the probability density function must be unity, the mean is zero, the variance is unity, the skewness is S and the kurtosis is K.

$$\begin{aligned}
A &= \int_{-\infty}^{\infty} f_Y(y)dy = 1 \\
\mu_Y &= \int_{-\infty}^{\infty} y f_Y(y)dy = 0 \\
\sigma_Y^2 &= \int_{-\infty}^{\infty} y^2 f_Y(y)dy = 1 \\
S^3 &= \int_{-\infty}^{\infty} y^3 f_Y(y)dy \\
K^4 &= \int_{-\infty}^{\infty} y^4 f_Y(y)dy
\end{aligned} \tag{2}$$

The first criterion is automatically met for a valid distribution. The third is a convenience. The realization can be rescaled later to the desired variance. In this implementation the integrals are approximated with sums.

Waveform Generation Using a ZMNL Function

The central idea is to find a function $y = g(x)$, which will convert a Gaussian waveform into a non-Gaussian waveform with a specified skewness and kurtosis. The ideas have been presented before, but not coupled with an algorithm to generate the Gaussian waveform with an arbitrary spectrum shape.

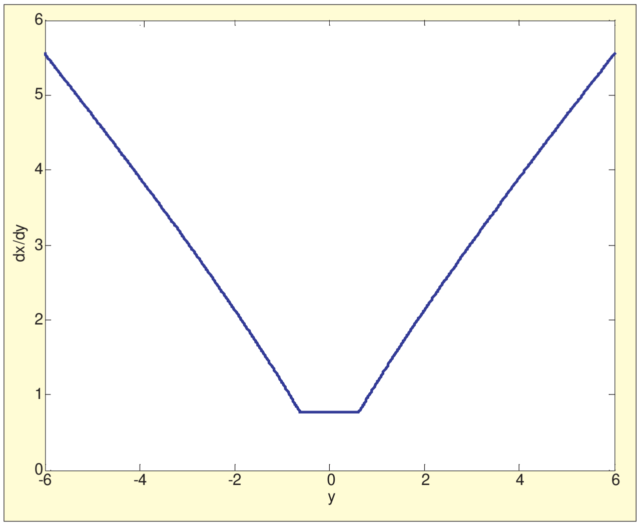

Figure 2. dx/dy, S = 0, K = 2.

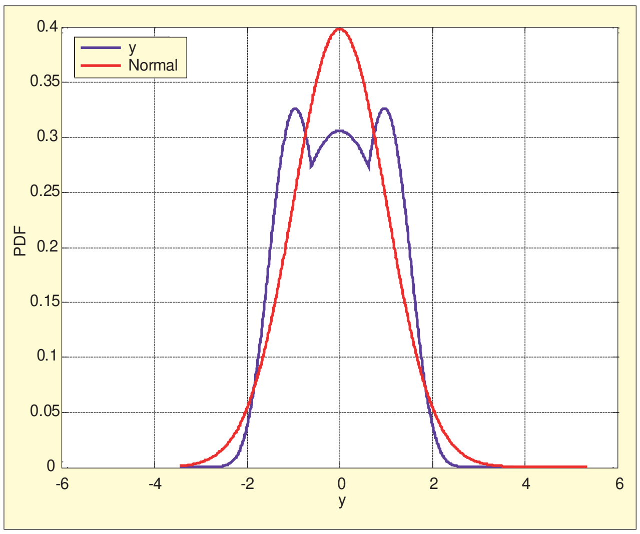

Figure 3. Probability density, S = 0, K = 2.

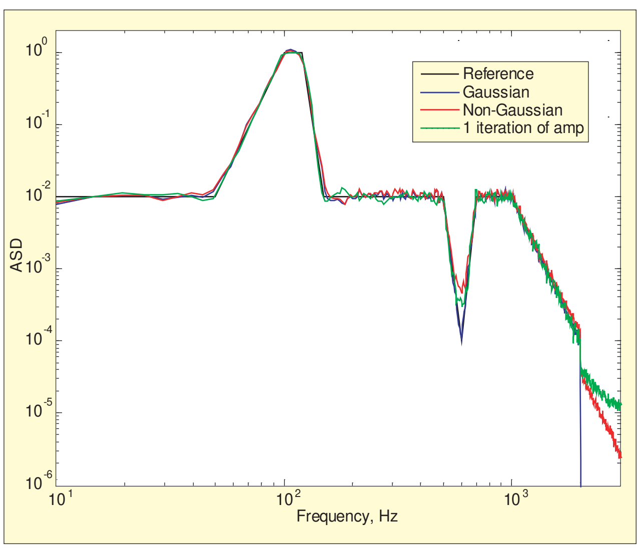

Figure 4. Autospectral densities, S = 0, K = 2.

In this article, the Gaussian waveform is generated using the same methods used for generating waveforms for random shaker testing. The waveforms are then distorted using the ZMNL function to create non-Gaussian waveforms. Three different methods are presented for generating the ZMNL function. Each has slight advantages and disadvantages. Each will cover a slightly different range of skewness and kurtosis. All

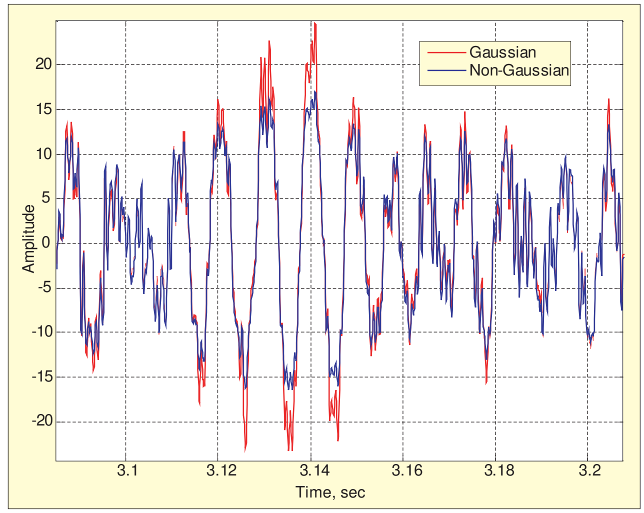

Figure 5. Small portion of time histories, S = 0, K = 2.

will give similar results.

Method 1. We would like the function $y = g(x)$ and its derivative, dy/dx, to be continuous. We will actually generate the inverse function $x = g^{-1}(y)$ and use a linear interpolation to generate the function values. The function chosen for this implementation is:

$$
\begin{aligned}
x(y) &= A\frac{a}{\alpha}\left(\frac{y^{\alpha}}{a^{\alpha}} + \alpha - 1\right) + AB \qquad && y > a \\
&= Ay + AB && -b \le y \le a \\
&= -A\left(\frac{b}{\beta}\right)\left(\frac{-(-y)^{\beta}}{b^{\beta}} - \beta + 1\right) + AB && y < -b
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
\frac{dx}{dy} &= Aa^{1-\alpha}y^{\alpha-1} \qquad && y > a \\
&= A && -b \le y \le a \\
&= -Ab^{1-\beta}(-y)^{\beta-1} && y < -b
\end{aligned} \tag{4}
$$

There are six parameters in the model (A, α, a, b, β, B), more than needed, which may lead to multiple solutions. However, the sixth parameter, B, was found to be necessary to force the mean to zero in some cases. B is the only parameter allowed to be negative. The function and its first derivative are continuous over the entire range of real values. The function is monotonically increasing. The function is linear for small values of y ($-b \le y \le a$).

The MATLAB® function **fminsearch** was used to find the parameters for a given skewness and kurtosis. The error function:

$$
E = |A - 1| + |\mu_y| + |\sigma_y - 1| + |S_y - S| + |K_y - K| \tag{5}
$$

was minimized. In general, solutions for negative skewness are not necessary. A time history with a positive skewness can always be inverted to achieve the same amplitude of negative skewness.

Method 2, Cubic Function. Other suitable functions can be found. Winterstein, *et al.*, used a cubic transformation and the inverse (which form depended on K):[6]

$$
\begin{aligned}
y &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 \qquad && K \ge 3 \\
x &= c_0 + c_1 y + c_2 y^2 + c_3 y^3 && K < 3
\end{aligned} \tag{6}
$$

To assure the function is monotonic, the derivative must be positive at zero and should have no real roots. This results in the following constraints:

$$
c_1 > 0, \quad c_3 > 0, \quad |c_2| < \sqrt{3c_1c_3} \tag{7}
$$

This requires use of a nonlinear optimization method with multiple inputs and constraints. The example here uses **fmincon** from the MATLAB optimization toolbox. The cubic function does have the advantage that the second derivative is

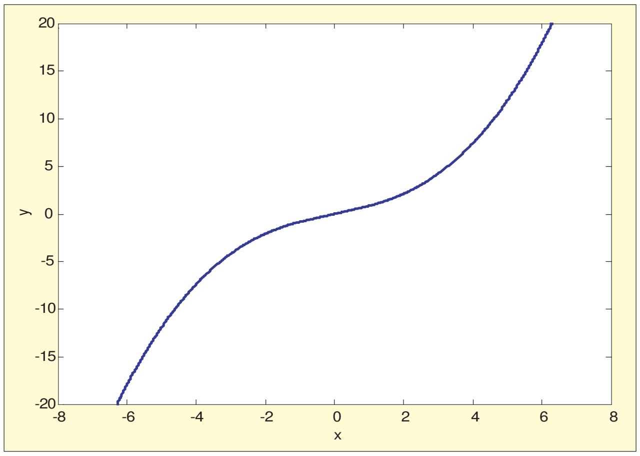

Figure 6. ZMNL function, S = 0, K = 6.

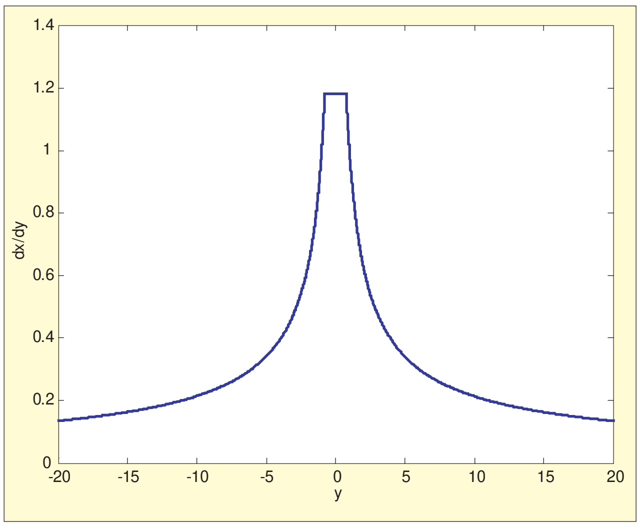

Figure 7. dx/dy, S = 0, K = 6.

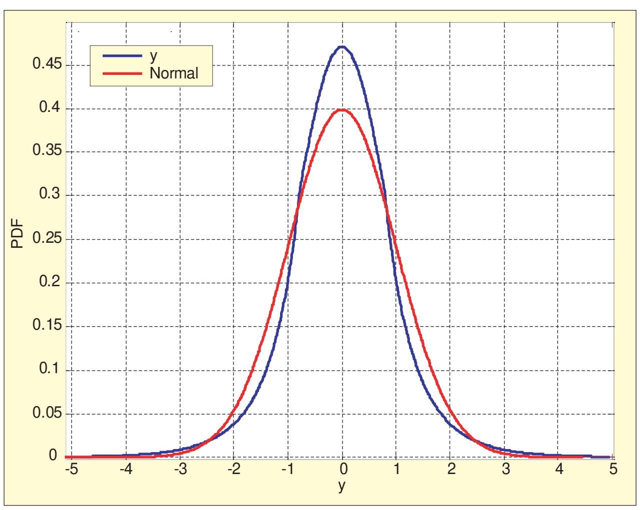

Figure 8. Probability density, S = 0, K = 6.

also continuous which results in a 'smoother' distribution. Winterstein, *et al.*, also considered using distributions other than Gaussian for X.[6]

Method 3, Hermite Polynomials. A closed-form solution for a cubic function is given by Winterstein[7] using Hermite polynomials for $K > 3$. In this formulation, x is the non-Gaussian waveform, and y is the Gaussian waveform.

$$y(x) = \left[\sqrt{\xi^2(x)+c} + \xi(x)\right]^{1/3} - \left[\sqrt{\xi^2(x)+c} - \xi(x)\right]^{1/3} - a \qquad (8)$$

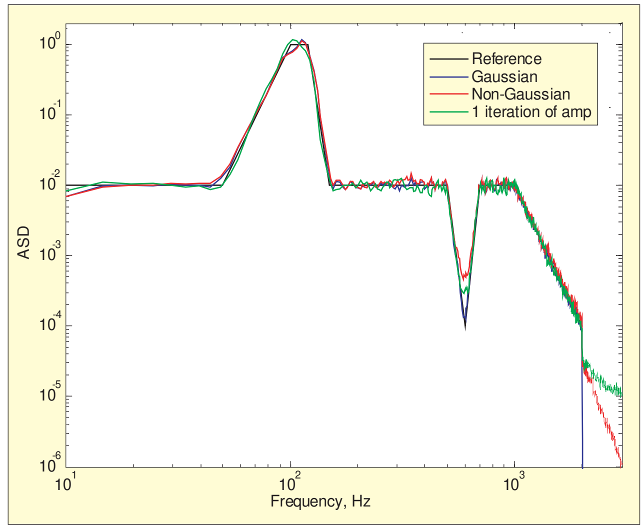

Figure 9. Autospectral densities, S = 0, K = 6.

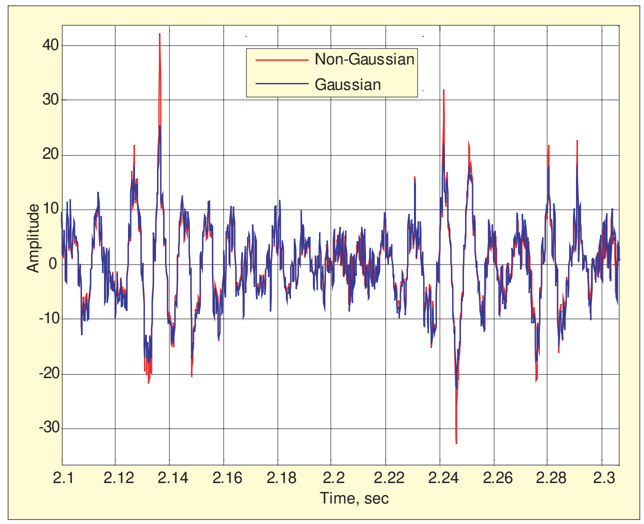

Figure 10. Small portion of time histories, S = 0, K = 6.

where:

$$\xi(x) = 1.5b\left(a + \frac{x}{\kappa}\right) - a^3 \ , \ a = \frac{\hat{h}_3}{3\hat{h}_4} \ , \ b = \frac{1}{3\hat{h}_4} \ , \ c = \left(b - 1 - a^2\right)^3$$

$$\hat{h}_3 = \frac{S}{4 + 2\sqrt{1 + 1.5(K-3)}} \ , \ \hat{h}_4 = \frac{\sqrt{1 + 1.5(K-3)} - 1}{18}$$

$$\kappa = \frac{1}{\sqrt{1 + 2\hat{h}_3^2 + 6\hat{h}_4^2}}$$

The positive roots are used. An improvement suggested by Gurley[8] uses the above solution as a starting point to solve the following pair of nonlinear equations:

$$S = \kappa^3\left(8\hat{h}_3^3 + 108\hat{h}_3\hat{h}_4^2 + 36\hat{h}_3\hat{h}_4 + 6\hat{h}_3\right) \qquad (9)$$

$$K = \kappa^4(60\hat{h}_3^4 + 3348\hat{h}_4^4 + 2232\hat{h}_3^2\hat{h}_4^2 + 60\hat{h}_3^2 + 252\hat{h}_4^2 + 1296\hat{h}_4^3 + 576\hat{h}_3^2\hat{h}_4 + 24\hat{h}_4 + 3) \qquad (10)$$

These solutions do not guarantee a monotonic function for all S and K. Generally, a kurtosis greater than three is associated with a softening nonlinearity (widening distribution tails), and a kurtosis less than three is associated with a hardening nonlinearity (narrowing distribution tails). The range of the skewness is limited by:[7]

$$S^2 \le -1 + K$$

The range of skewness and kurtosis for a unimodal probability density function is much less[7] than that indicated above. A closed-form solution for a cubic function is given by

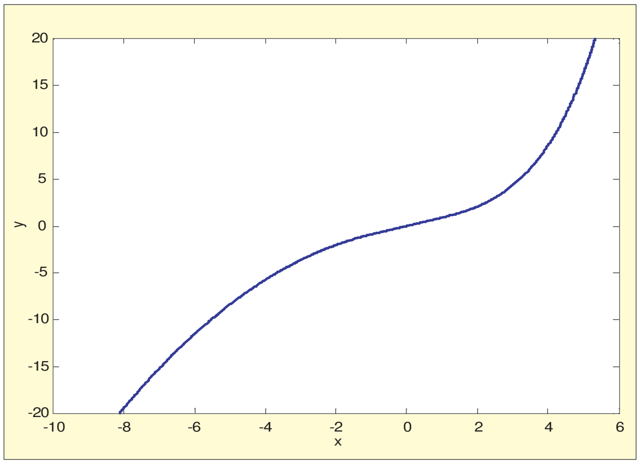

Figure 11. ZMNL function, S = 0.3, K = 5.5.

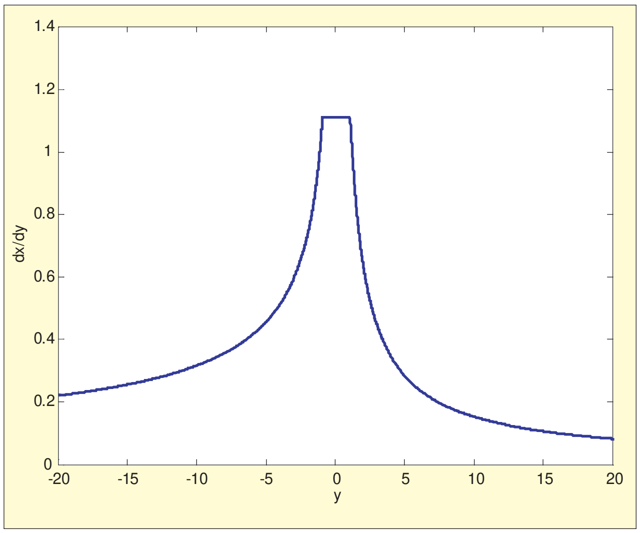

Figure 12. dx/dy, S = 0.3, K = 5.5.

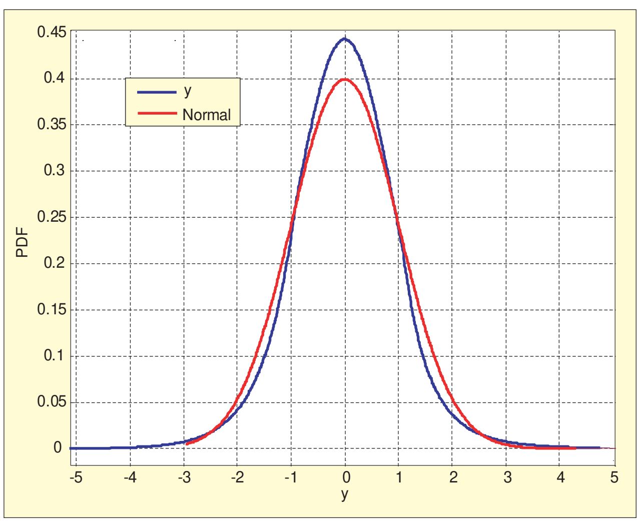

Figure 13. Probability density, S = 0.3, K = 5.5.

Winterstein[7] using Hermite polynomials for $K < 3$.

$$x(y) = y - \frac{S}{6}\left(y^2 - 1\right) - \frac{K-3}{24}\left(y^3 - 3y\right) \tag{11}$$

As above, this formulation will work only for a limited range of S and K.

Examples

This section illustrates several examples. We will start with Method 1 and compute realizations for skewness of zero with

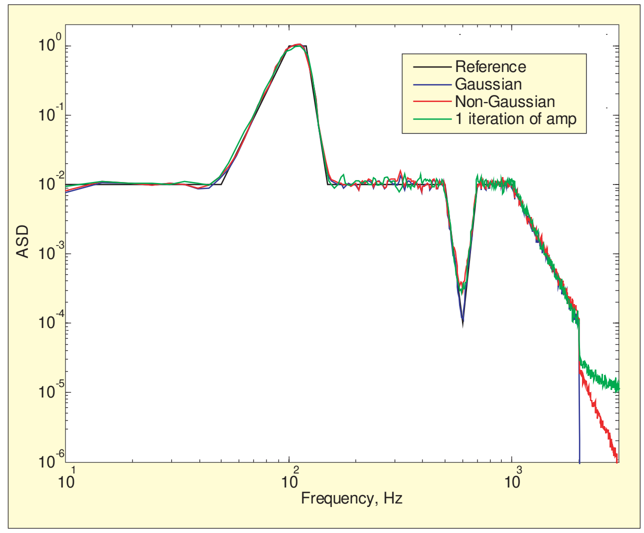

Figure 14. Autospectral densities, S = 0.3, K = 5.5.

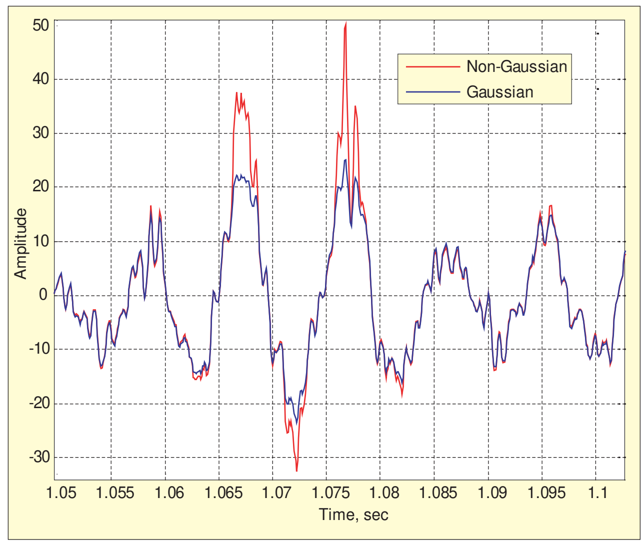

Figure 15. Small portion of time histories, S = 0.3, K = 5.5.

two values of kurtosis, one less than three, and one greater than three. Remember a skewness of zero and a kurtosis of three is a Gaussian waveform. We will then increase the skewness and compute an example. Method 2 is then used and compared to Method 1 with zero skewness and a kurtosis less than three. Method 3 is then compared to Method 1 for a nonzero skewness and a kurtosis greater than three. In each case, the ZMNL function, dx/dy, the PDF (probability density function), the autospectral density, the Gaussian time history and the non-Gaussian time history will be plotted.

S = 0, K = 2, Method 1. The parameters needed to achieve a skewness of 0 and a kurtosis of two are:

$$A = 0.7672,\ \alpha = 1.87,\ a = 0.618,\ b = 0.618,\ \beta = 1.87,\ B = 0$$

Note that the exponents are greater than one, which will generally be the case for a small skewness and a kurtosis less than three. The resulting ZMNL function is shown in Figure 1. In all the examples, the ZMNL has been normalized for a standard deviation of one. The derivative dx/dy and the resulting PDF are shown in Figures 2 and 3. A Gaussian PDF is also shown for reference. Note that a trimodal PDF is generated by the requirement to suppress large values of y to achieve a small kurtosis. (The kurtosis of a Gaussian waveform is three.)

A realization of 100,000 points was created using the target spectrum shown in Figure 4. The frequency-domain method discussed by Smallwood and Paez[9] was used to generate the time histories. This is basically the same method used to generate time histories for random vibration control systems.[10] A

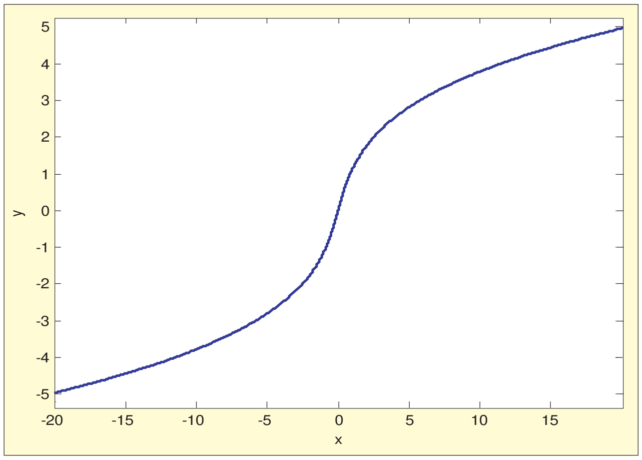

Figure 16. ZMNL function, S = 0, K = 2, using the cubic function.

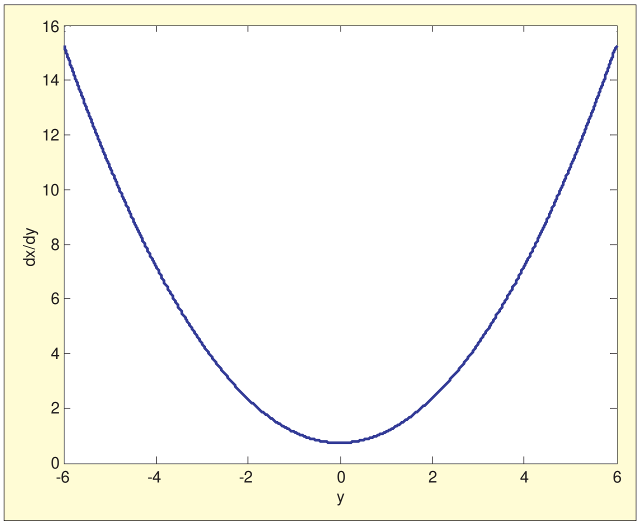

Figure 17. dx/dy, S = 0, K = 2, using a cubic function.

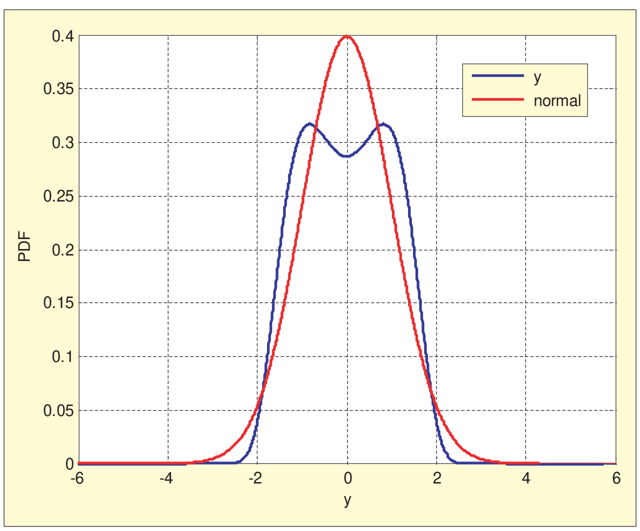

Figure 18. Probability density, S = 0, K = 2, using a cubic function.

Hanning window with 75% overlap was used. The sample rate was 10,000, and the block size was 2048. This method is basically the inverse of the Welch method for estimating an autospectral density. After the autospectral density (ASD) of y was estimated, a single iteration of the spectrum of x was performed to correct the ASD of x in an attempt to improve the ASD of y. The resulting ASD is plotted on Figure 4. Also shown in Figure 4 are the estimated ASD of the Gaussian waveform, x, and the non-Gaussian waveform, y. A small portion of the time histories is shown in Figure 5. Note how the large values

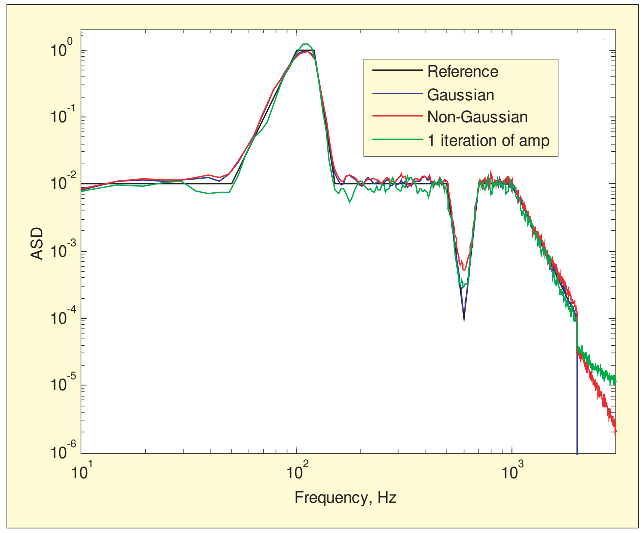

Figure 19. Autospectral densities, S = 0, K = 2, using a cubic function.

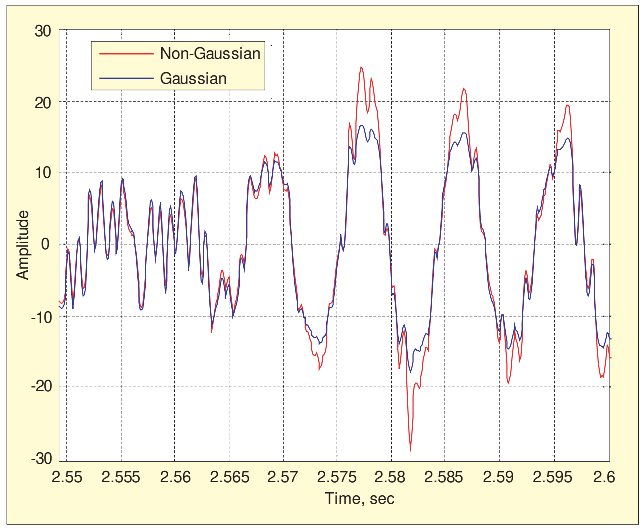

Figure 20. Small portion of time histories, S = 0, K = 2, using a cubic function.

of x have been suppressed in y. The 'whitening' effect is seen mainly in the notch and the roll-off at high frequencies.

S = 0, K = 6, Method 1. For this example, the skewness was kept at zero, and the kurtosis was increased to six. The same target ASD in the previous example was used. A realization of 100,000 points was generated as in the previous example. The parameters for this example are:

$$A = 1.1809,\ \alpha = 0.332,\ a = 0.779,\ b = 0.779,\ \beta = 0.332,\ B = 0$$

In this case, the exponents are less than one. Figures 6 through 10 show the same information as the previous example. Notice that now the peaks in y are exaggerated versions of the peaks in the Gaussian x. A slight excess in the ASD of y is seen near 300 Hz. This is the third harmonic of the peak in the ASD at 100 Hz. As in the previous example, the largest deviations from the reference spectrum are at the ASD minima.

S = 0.3, K = 5.5, Method 1. This is an example of a nonsymmetric waveform. The parameters are:

$$A = 1.1803,\ \alpha = 0.1113,\ a = 1.0855,\ b = 0.9118,$$
$$\beta = 0.4766,\ B = 0.0043$$

The same ASD in the previous examples is used. As before, a realization of 100,000 points was generated. Note that α is smaller than β, which suggests a positive skewness. The results are shown in Figures 11 through 15.

S = 0, K = 2, Method 2, Using a Cubic Function. The same spectrum was used in the first example. In this case, the cubic function was:

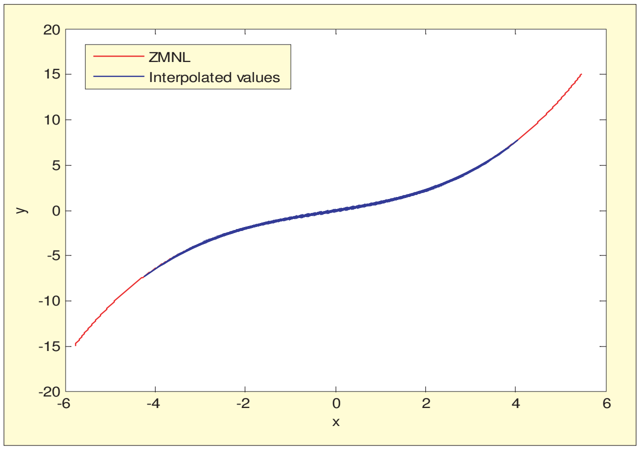

Figure 21. ZMNL function, S = 0.3, K = 5.5.

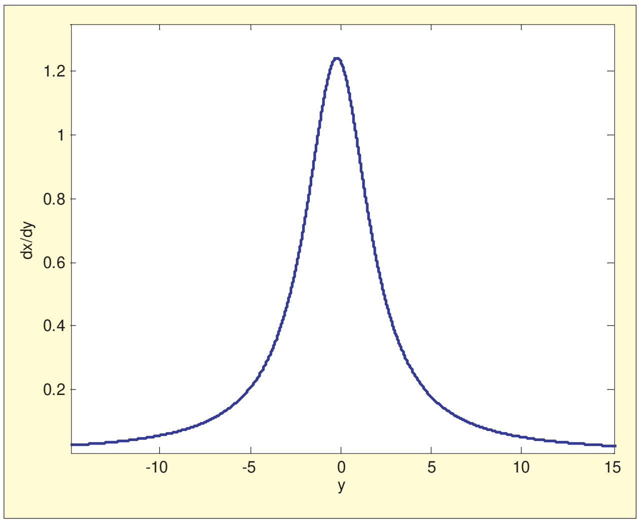

Figure 22. dx/dy, S = 0.3, K = 5.5.

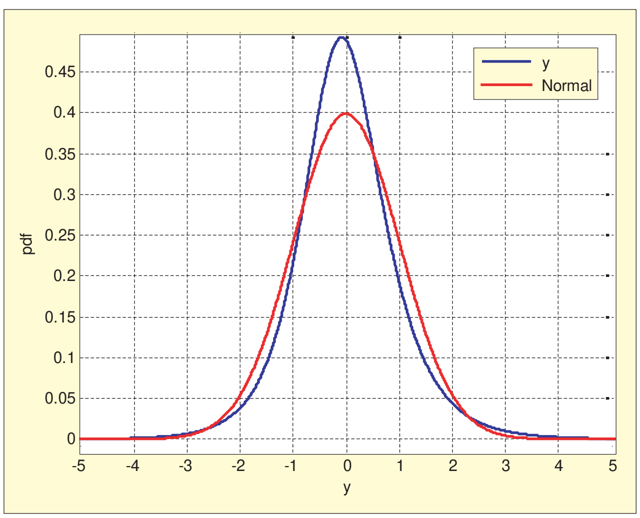

Figure 23. Probability density, S = 0.3, K = 5.5.

$$x = c_0 + c_1 y + c_2 y^2 + c_3 y^3$$

with the parameters:

$$c_0 = 0,\ c_1 = 0.71949,\ c_2 = 0,\ c_3 = 0.13447$$

The results are shown as Figures 16 through 20. A comparison to Figures 1 through 5 shows that the two ZMNL functions give very similar results. This example does illustrate that the solution is not unique. Many functions can be found that will result in a transformation with the same skewness and kurtosis.

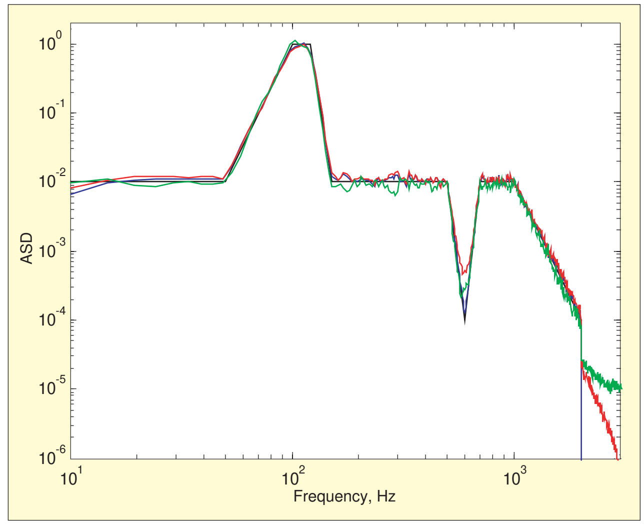

Figure 24. Autospectral densities, S = 0.3, K = 5.5.

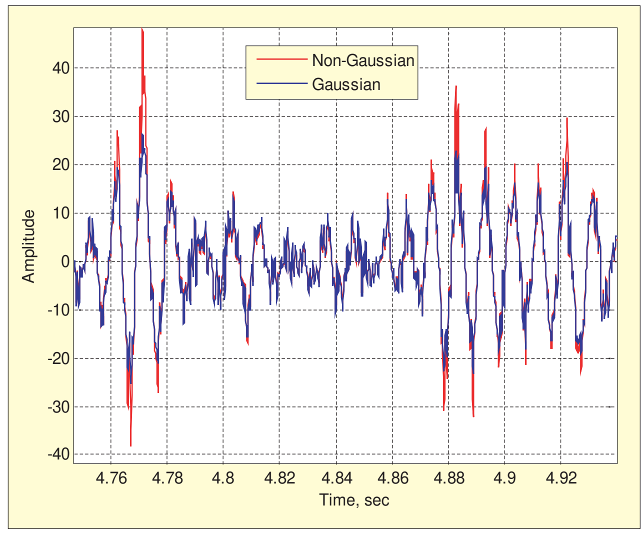

Figure 25. Small portion of time histories, S = 0.3, K = 5.5.

S = 0.3, K = 5.5, Method 2. This is the same as one of the previous examples, except Method 3 was used. The parameters are:

$$c_0 = -0.035872,\ c_1 = 0.8121,\ c_2 = 0.036006,\ c_3 = 0.058728$$

This is an example of a nonsymmetric waveform. The same ASD from the previous examples is used. As before, a realization of 100,000 points was generated. The results are shown in Figures 21 through 25. The results are very similar to the previous example. Figure 24 shows that the iteration energy is removed from the Gaussian x near the minima to correct the spectrum of y with some success. Method 3 results in essentially identical results for this example.

Conclusions

Three simple but effective methods for generating realizations of a stationary random process with specified skewness and kurtosis using a zero-memory nonlinear (ZMNL) function are presented. They all give similar, but not identical results. Each covers similar but slightly different ranges of skewness and kurtosis. Each will result in slightly different distributions. All the methods are relatively easy to implement using modern computational tools such as MATLAB®. MATLAB is available from The MathWorks, www.mathworks.com.

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MATLAB scripts for generating non-Gaussian vibration test signals using the methods presented in this article are available from the author: dsmallwood@comcast.net.