Shock Response Spectrum – A Primer

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The intent of this paper is to provide a basic overview, or primer, of the shock response spectrum (SRS). The intended audience for this paper is the design engineers who have need to work with the shock response spectrum, and would like to understand the underlying detail. The shock response spectrum was first conceived by Dr. Maurice Biot and is described in his Ph.D. thesis published in 1932. Hence, the SRS has been in existence for a long time. It has been used to characterize the frequency response of shock environments to estimate the maximum dynamic response of structures. The SRS is commonly used to characterize the frequency content of an acceleration time-history record.

Nomenclature

SRS = Shock response spectrum
DOF = Degree of freedom
SDOF = Single degree of freedom
MDOF = Multi-degree of freedom
m = Mass of a SDOF system
m_i i^{th} mass of a MDOF system
k = Spring stiffness of a SDOF system
$x(t)$ = Absolute displacement of mass of a SDOF system as a function of time
$x_i(t)$ = Modal coordinate displacement for the i^{th} mode of vibration
$u_i(t)$ = Absolute displacement of i^{th} mass of a MDOF system as a function of time
$z(t)$ = Displacement of a SDOF system mass relative to its' base as function of time
$z_j(t)$ = Displacement of the j^{th} mass of a MDOF system relative to its' base as a function of time
$\ddot{u}_b(t)$ = Base acceleration of a SDOF or MDOF system as a function of time
ω = Frequency of a one-DOF system
S_D = Spectral displacement
S_V = Spectral velocity
S_A = Spectral acceleration
$[M]$ = Mass matrix of a MDOF system
$[K]$ = Stiffness matrix of a MDOF system
ω_i = Frequency of the i^{th} mode of vibration of a MDOF system
$\{\phi\}_i$ = Eigen vector or mode shape for the i^{th} mode of vibration of a MDOF system
P_i = Participation factor for the i^{th} mode of vibration of a MDOF system
$\bar{M}_i$ = Modal mass for the i^{th} mode of vibration of a MDOF system
$\{\ddot{u}(t)\}$ = Absolute acceleration vector for a MDOF system
$\{\ddot{z}(t)\}$ = Relative acceleration vector for a MDOF system
$\{x(t)\}$ = Modal coordinate displacement vector
SAVIAC = Shock and Vibration Information Analysis Center

The organization of this paper is as follows:
- Definitions and examples of shock and the shock response spectrum.
- Typical events characterized by a shock response spectrum
- Examples of design spectra and how they are determined.
- How a shock response spectrum can be used in the analysis of linear multi-DOF (degree of freedom) systems.
- The naval shock design spectrum including the spectrum dip phenomenon.

Before addressing the details of the shock response spectrum, a few definitions and examples may be helpful.

Shock. Below are several definitions of "shock" from various sources:

At the first Shock and Vibration Symposium in 1947, mechanical shock was defined as "a sudden and violent change in the state of motion of the component parts or particles of a body or medium resulting from the sudden application of a relatively large external force, such as a blow or impact."

A mechanical or physical shock is a sudden acceleration or deceleration caused, for example, by impact, drop, earthquake, or explosion. Shock is a transient physical excitation. (*Wikipedia*)

Mechanical shock may be defined as a sudden change in velocity and is a major design consideration for a wide variety of systems and their components. (SAVIAC's Mechanical Shock Test Techniques & Data Analysis course description).

Shock Response Spectrum. Similarly, several definitions for the shock response spectrum are:

The shock response spectrum is a graphical representation of an arbitrary transient acceleration input, such as shock in terms of how a single degree of freedom (SDOF) system (like a mass on a spring) responds to that input. Actually, it shows the peak acceleration response of an infinite number of SDOFs, each of which has different natural frequencies. (*Wikipedia*)

Shock response spectrum analysis is the maximum response of a series of single degree of freedom systems [having the] same damping to a given transient signal.

Transient Shock Examples. The following illustrations show examples of transient shock. Figure 1 is the acceleration-time history of the east-west component of the well known El Centro earthquake. Figure 2 is a photograph of the damage that was done by this earthquake. Ballistic shock to a ground combat vehicle and shock to shipboard equipment from a near miss underwater explosion is depicted in the photographs of Figures 3 and 4, respectively.

A shock response spectrum transforms the acceleration-time history from the time domain to the maximum response of a SDOF system in the frequency domain. Figure 5 is the shock response spectrum of the El Centro acceleration-time history of Figure 1. It represents how a series of linear single degree of freedom systems at various frequencies would respond to this shock transient. The SRS values are the maximum absolute accelerations that the SDOF system tuned to each frequency would experience over the entire transient. Notice that the spectral value at high frequencies converges to approximately 210 cm/sec^2, which corresponds to the maximum ground acceleration from Figure 1 at approximately 22 sec. The reason for this will be discussed later.

Maurice Biot. Dr. Maurice Biot (Figure 6) conceived the shock response spectrum as documented in his 1932 Ph.D. thesis. He defined the SRS as the maximum response motion from a set of single DOF oscillators covering the frequency range. Biot showed how to pick a small number of modes which are adequate for design. For earthquake applications, he used the traditional assumption that the ground's motion is not affected by the dynamic motion of the building. Later study demonstrated that this assumption is overly conservative and leads to over design of equipment, which is especially evident in the case of shipboard mounted equipment subjected to naval shock from near-miss underwater explosions.

Figure 7 is a qualitative graphical representation on the construction of a shock response spectrum. Consider a transient shock acceleration-time history input applied to a base where a series of single DOF linear oscillators of different frequencies are mounted. Also, assume that the base is sufficiently massive such that the motions of the oscillators do not affect the motion of the base. The lowest frequency system (frequency f_1) is on the left hand side of the base and the highest frequency system (frequency f_6) is on the right hand side. The transient acceleration response of each oscil-

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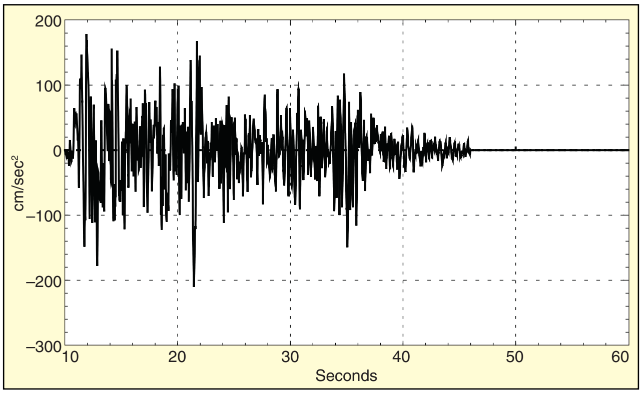

Figure 1. El Centro E-W ground acceleration.

Figure 2. Earthquake shock damage.

Figure 3. Ballistic shock.

Figure 4. Naval shock.

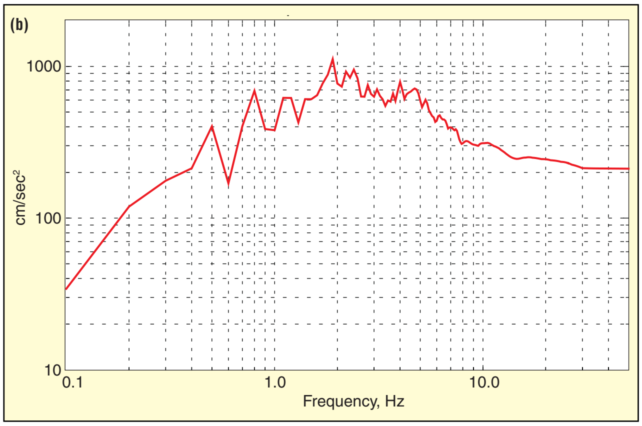

Figure 5. Acceleration-time history of Figure 1 transformed to a shock response spectrum.

Figure 6. Maurice Biot.

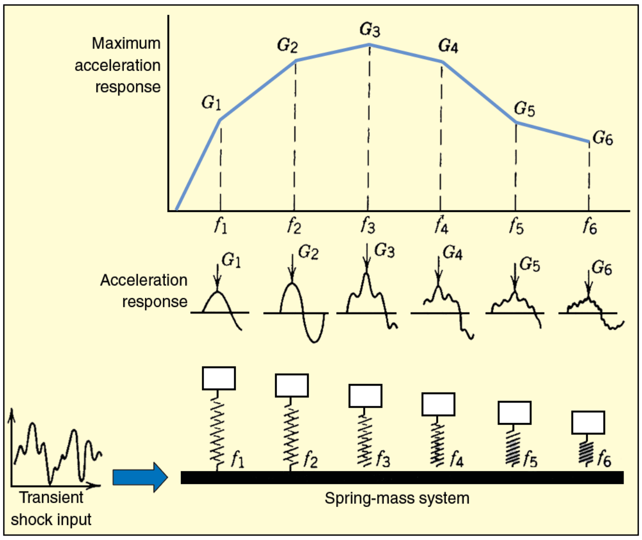

Figure 7. How a shock response spectrum is developed.[3]

lator is illustrated graphically above each oscillator.

The maximum acceleration response of the oscillator 1 is G_1. Note that the acceleration response for oscillator 1 has the lowest frequency and lower maximum acceleration amplitude than the next few oscillators immediately to the right. Oscillator 2 has a higher frequency and maximum amplitude G_2. Oscillator 3 has the highest maximum response G_3 relative to the others. Progressing to the right, the frequency continues to increase but the maximum acceleration response drops off. The maximum acceleration G_i response of each oscillator is plotted as a function of frequency at the top of the figure. This is a graphical representation of the shock response spectrum for the system.

The SRS graph at the top of Figure 7 is used to characterize the frequency content of the transient shock input signal. Oscillator 3, for example, has the greatest maximum acceleration response to

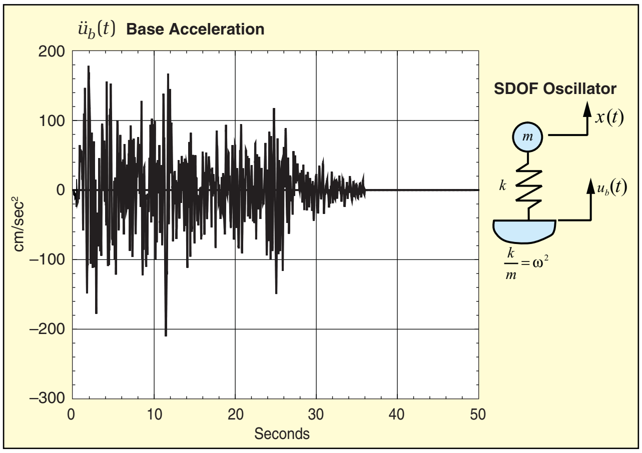

Figure 8. Base acceleration SDOF oscillator.

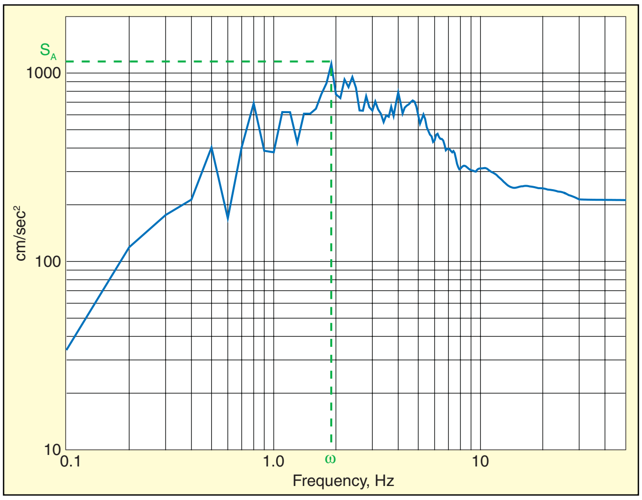

Figure 9. Shock response spectrum.

the transient input. The maximum response drops off for systems of higher and lower frequencies, representing stiffer and softer systems, respectively.

Figure 7 is a graphical representation of a SRS. Figures 8 and 9, along with the equations that follow are a mathematical development of a shock response spectrum. Figure 8 depicts the application of an acceleration time-history to the base of a linear, non-damped single degree of freedom system. Figure 9 is the shock response spectrum of the acceleration-time history of Figure 8. The absolute coordinate of the mass in Figure 8 is given by $x(t)$. The equation of motion of the SDOF system is determined by a balance of the spring and inertial forces acting on the mass:

$$m\ddot{x}(t) + k\left(x(t) - u_b(t)\right) = 0 \qquad (1)$$

A relative coordinate $z(t)$ (meaning relative to the base motion) can be defined as:

$$z(t) \equiv x(t) - u_b(t) \qquad (2)$$

Differentiating (2) twice and substituting into (1) yields the equation of motion in relative coordinates:

$$\ddot{z}(t) + \omega^2 z(t) = -\ddot{u}_b(t) \qquad (3)$$

From substitution of (2) into (1), it is found that the absolute acceleration of the mass is proportional to the relative displacement:

$$\ddot{x}(t) = -\omega^2 z(t) \qquad (4)$$

where ω^2 is k/m. The closed form solution to (3) is obtained from Duhamel's Integral (5). Unless the base acceleration is a relatively simple function that can be integrated after being multiplied by sin $\omega(t{-}\tau)$, (3) is typically solved by numerical integration:

$$z(t) = -\frac{1}{\omega}\int_0^t \ddot{u}_b(\tau)\sin\omega(t-\tau)d\tau \qquad (5)$$

Using the results from (4), the absolute acceleration of the mass is:

$$\ddot{x}(t) = \omega\int_0^t \ddot{u}_b(\tau)\sin\omega(t-\tau)d\tau \qquad (6)$$

The shock response spectrum is defined as the maximum $|\ddot{x}(t)|$ for each frequency:

$$S_A \equiv |\ddot{x}(t)|_{\max} \qquad (7)$$

Substituting (6) into (7), the SRS is given by (8). Since this example did not include damping, this is the non-damped spectrum. When damping is included, it can be shown that the damped SRS is given by (9):

$$S_A = \left|\omega\int_0^t \ddot{u}_b(\tau)\sin\omega(t-\tau)d\tau\right|_{\max} \quad \text{(No damping)} \qquad (8)$$

$$S_A = \left|\omega\int_0^t \ddot{u}_b(\tau)e^{-\xi\omega(t-\tau)}\sin\omega(t-\tau)d\tau\right|_{\max} \quad \text{(With damping)} \qquad (9)$$

Maximax, Primary and Residual Shock Spectrum[1]

Figure 10 is an illustration of transient system amplitude as a function of time. There are a number of different shock response spectra that can be constructed from a single transient response depending on which feature of the transient is selected. The primary region corresponds to the time that the shock excitation is applied to the system, and the residual is the region of the response after the excitation has ended. Point 1 on the figure, for example, corresponds to three characteristics. Point 1 is the maximum absolute response of the system over the entire duration (maximax), the maximum negative response of the system over the entire duration (maximum negative) and finally the maximum negative response during the primary phase (primary negative). A spectrum plot termed the maximax spectrum, for example, would be a plot of the maximax points over all system frequencies. Similarly, spectrum plots for the maximum negative and primary negative could be developed based on the same approach.

The other points on Figure 10 are given similar terms. Point 2 is maximum response during the primary phase of the response (primary positive). Points 3 and 4 occur during the residual portion of the response. Point 3 is the maximum positive response during the entire duration (maximum positive) and is also the maximum positive response during the residual portion (residual positive). Point 4 is the maximum negative response during the residual portion (residual negative).

To summarize:

- The maximax spectrum consists of the maximum absolute response recorded as a function of the system natural frequency.
- The maximum positive spectrum contains only the maximum positive response.
- The maximum negative spectrum contains only the maximum negative response.
- The primary spectrum is made up of the maximum absolute responses during the excitation.
- The residual spectrum contains the peak response occurring after the excitation has completely decayed.

Events Characterized by a Shock Response Spectrum

Earthquakes, shipboard naval shock due to near-miss underwater explosions, and pyroshock are typical events characterized by a shock response spectrum.

An *earthquake* results from a sudden release of energy in the Earth's crust that creates seismic waves. At the Earth's surface, earthquakes manifest themselves by a shaking and sometimes displacement of the ground. The shaking in earthquakes can also trigger landslides and occasionally volcanic activity. Earthquakes are caused primarily by rupture of geological faults.

The elemental source of *(naval) shock* is the sudden application of external forces to a portion of the ship's hull. The term "shock" is applied to the sudden, transient motion of an item of machinery as transmitted by the foundation or the mounting from the ship's structure. Thus the term "shock" is employed in a relatively restricted sense, and does not include the destruction of the ship's

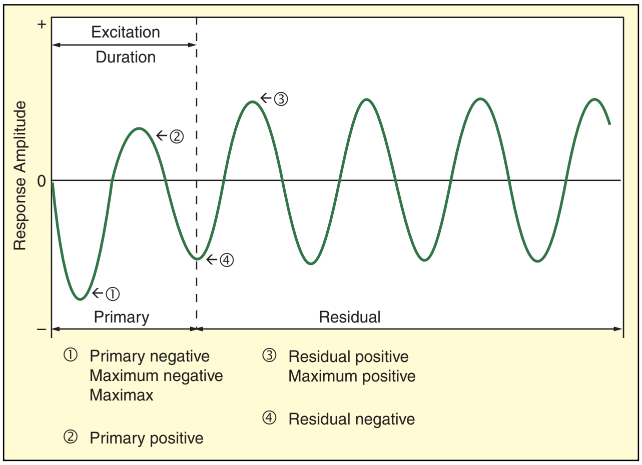

Figure 10. Response maxima.

Figure 11. Earthquake damage to apartment building in San Francisco's Marina District.

Figure 12. Ship shock trials.

structure by means of direct exposure to an explosion, or the damage to equipment as the result of collision of a projectile or other object, or damage due to extreme distortion of the foundation.[8]

Pyroshock refers to short-duration, high-amplitude, high-frequency, transient structural responses in aerospace vehicles. Pyroshock on rocket or missile systems is attributable to explosive bolts and nuts, pin pullers, separation of spent rocket booster stages, linear cutting of the structure, and other actions that produce a near-instantaneous release of strain energy. To support structural analysis of typical military and aerospace systems, a 20,000 Hz frequency response is always more than adequate.

Figure 11 shows the damage done to a building in the Marina district of San Francisco from the aftermath of an earthquake. Soil motion due to an earthquake is typically more horizontal than vertical, resulting in the lateral shear-like failures of the building

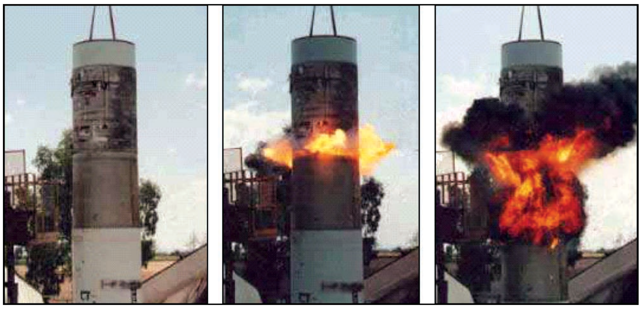

Figure 13. Pyroshock from launcher vehicle stage separation.

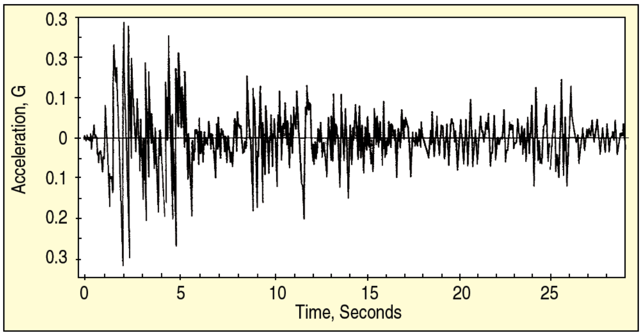

Figure 14. Accelerogram from May 18, 1940, El Centro earthquake.

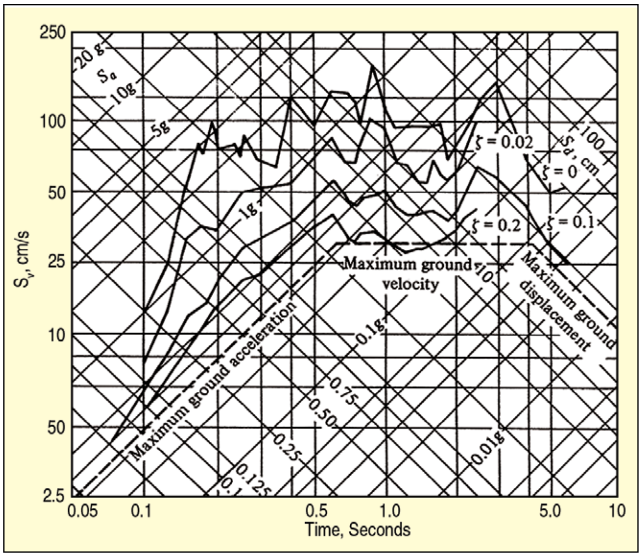

Figure 15. Response spectra from El Centro quake.

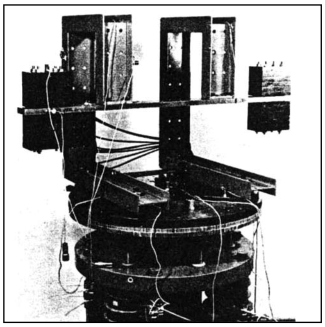

Figure 16. Naval shock test fixture.

in Figure 11. Figures 12 and 13 are photographs of the short duration events that correspond to shipboard shock and pyroshock, respectively.

Earthquake Spectrum. Figure 14 is the acceleration time-history record of the well known earthquake near El Centro California in 1940. This particular earthquake has been often cited and used for numerous studies. From the time-history, note that the maximum absolute value of the ground acceleration is approximately 0.3 g occurring approximately 2 sec into the transient.

The graph of Figure 15 is the shock response spectra for this earthquake. Note that in this case, the abscissa is not frequency, but rather the system period. Hence, frequency increases to the left in this case. There are several things to note about this figure. The spectra are plotted on a four coordinate, tripartite grid. Using this plotting technique, the spectral displacement, velocity and

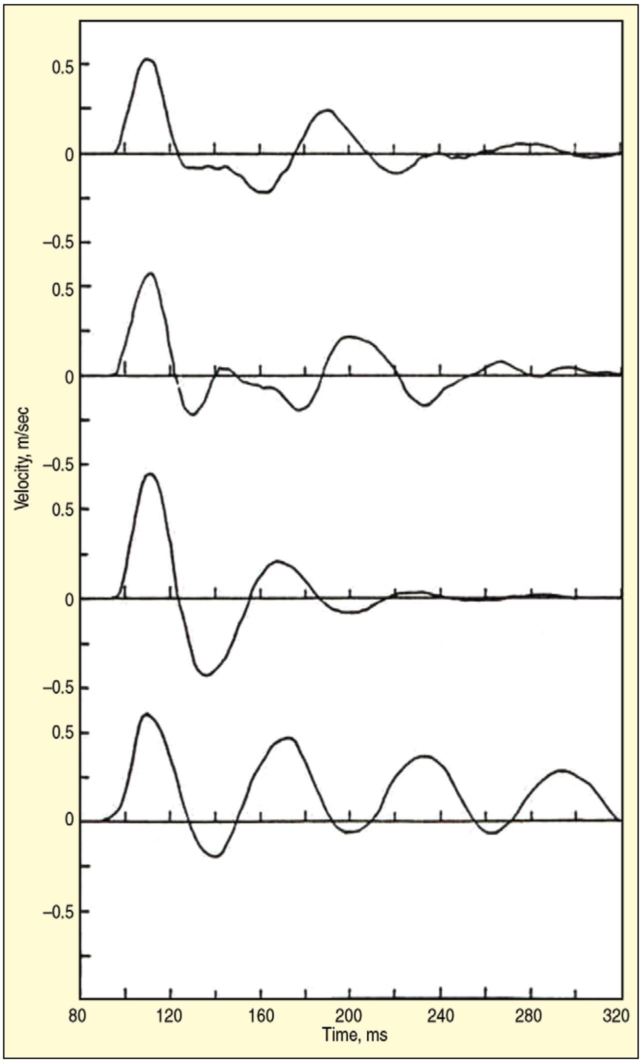

Figure 17. Velocity time histories.

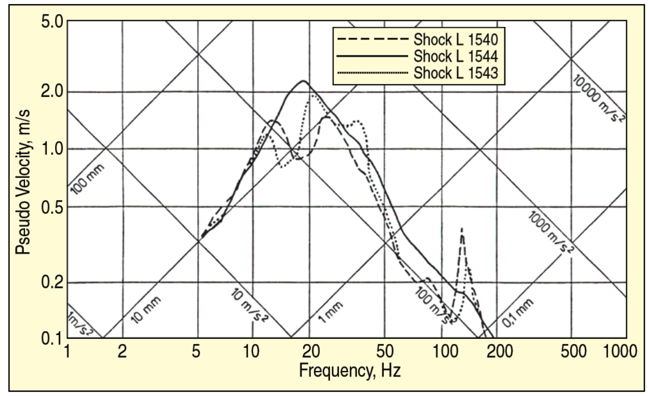

Figure 18. Response spectra for multiple shocks.

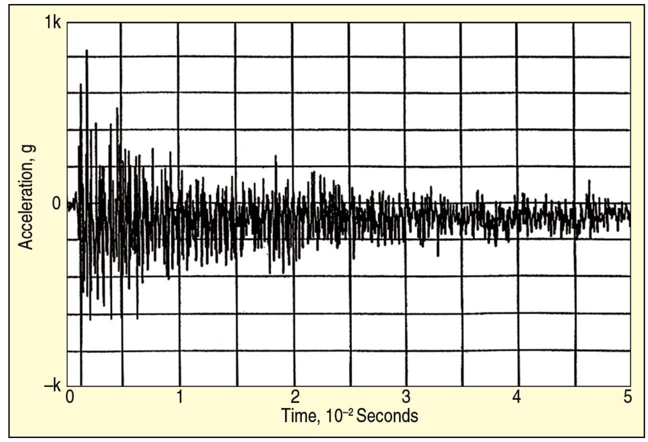

Figure 19. Acceleration time history.

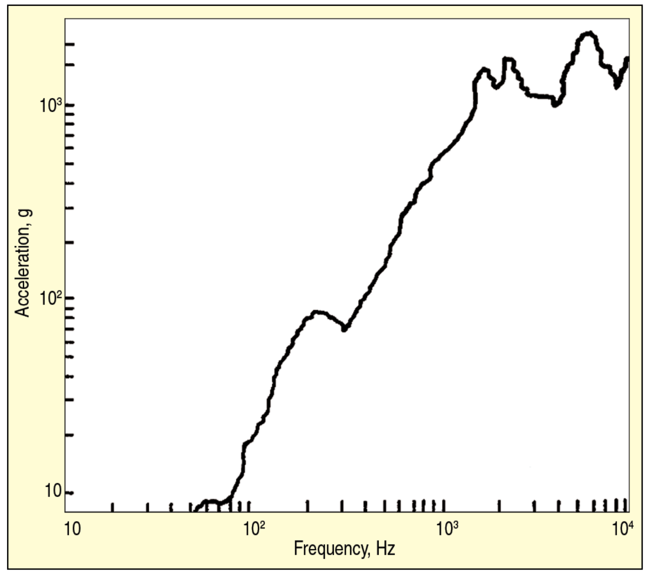

Figure 20. Shock response spectrum.

acceleration are all shown on the same graph. The grid lines with the positive 45 degree slope are measures of spectral acceleration, S_A in g. The horizontal grid lines are measures of spectral velocity, S_V in cm/sec. The grid lines with a negative slope of –45 degrees are a measure of spectra displacement, S_D in cm. Several solid curves are plotted on this graph where each corresponds to the spectral values for different amounts of damping. The percent of critical damping for each are listed by the corresponding graph. Notice also that the maximum ground motions (i.e., maximum base acceleration, velocity and displacement) are also plotted on the graph with dotted lines. Since the spectral lines are above the maximum ground base motions in the middle region of the graph, these represent regions of amplified response. For example, the maximum spectral acceleration is approximately 3.5 g, which is an order of magnitude greater than the aforementioned maximum ground acceleration. Observe that as the period decreases (or the frequency increases) the spectral accelerations begin to converge to the maximum ground acceleration of approximately 0.3 g. The reason for this will be discussed later.

Naval Shock. Figures 16 through 18 show the results of a test documented by Regoord[2] relative to shock testing associated with the Netherlands naval shipbuilding specifications. Figure 16 is a photograph of the test fixture used. Figure 17 shows velocity time-history plots for three different test configurations, depending on where the masses were mounted to the test fixture. Shock loading was applied to the underside of the fixture by a light weight shock testing machine. The shock response spectra from the three test configurations are plotted on Figure 18.

The intent of showing this test results is not to enlighten the reader about the Netherlands' shock specifications, but rather to illustrate in general the shock levels of shipboard shock as compared with that of earthquakes. For the El Centro earthquake, the shock duration is on the order of 10s of seconds, where naval shock shown on Figure 17 is a much shorter duration on the order of 100s of milliseconds. As expected, the shock levels for naval shock are much greater than for earthquakes. In the case of the earthquake, the maximum spectral acceleration is 3.5 g. The maximum acceleration of the naval shock response spectrum shown in Figure 18, is approximately 300 m/sec^2 which is about 30 g; one order of magnitude higher than earthquake shock in this example.

Pyrotechnic Shock. Figures 19 and 20 are typical durations and shock levels for pyrotechnic shock.[3] From the time-history plot, the duration of the shock loading is on the order of 10s of milliseconds and from the shock response spectrum the spectral acceleration is approximately 2,500 g. It is noted, however, that the SRS of Figure

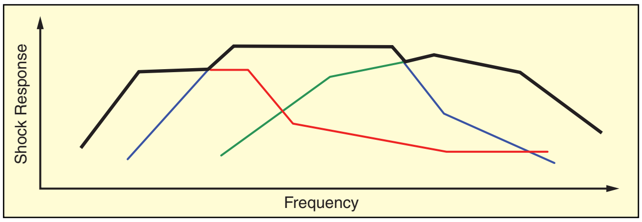

Figure 21. Maximum SRS envelope of three different spectra.

20 does not correspond to the specific time history of Figure 19.

Design Spectra

Early work done by Biot on seismology found that the damage potential of a single earthquake varied greatly from one building to the next. He noted that frequency peaks in shock spectra from a single earthquake are not constant within the same neighborhood. This lead to Biot recommending an envelope approach of all spectra for design purposes. Figure 21 is a schematic that illustrates how a maximum envelope is constructed. For this example, three different spectra are superimposed and the maximum envelope is constructed from the maxima of the three. Similarly, a minimum envelope can be constructed from the minima of the three spectra shown in the figure.

Relative to loadings on buildings, Biot found that stresses calculated using the maximum envelope approach were much higher than those observed from an actual earthquake. Biot attributed this to factors such as damping, plastic deformation, and possible interaction of nearby soil with the foundation of the building.

Housner Design Spectra. Housner[4] developed the first spectra used for seismic design of structures in the late 1950s. These were obtained by averaging and smoothing the response spectra from eight ground motion records, two from each of the following four earthquakes as shown in Figure 22.

- El Centro (1934)
- El Centro (1940)
- Olympia (1949)
- Tekiachapi (1952)

Newmark Design Spectra.[5,6] In the 1970s Newmark developed an earthquake design spectrum approach based on amplification factors applied to maximum ground motions. Figures 23 and 24 illustrate this technique. Figure 25 and Table 1 show the Newmark Design Spectrum for a specific soil condition. The amplification factors in Table 1 are listed for different probabilities of occurrence and also for various levels of damping of the structure. Figures 23 and 24 show how a spectrum is developed from the ground motion maxima. The region of amplified response is between the relatively high and relatively low frequency extremes of the spectrum. Note in Figure 23 at relatively high frequencies, the shock spectrum level approaches the maximum ground acceleration. This is the aforementioned feature that was observed in the earlier El Centro earthquake example from Figures 1 and 5.

The explanation for behavior can be aided by referring back to the single DOF oscillator in Figure 8. When the ground motion is applied to the base of this oscillator, the acceleration of the mass in general will be amplified as shown in the middle region for Figure 23. However, if the oscillator has a very high frequency, as characterized by a relatively stiff spring with respect to the mass, it will tend to respond like a rigid body relative to the base. As such, the spring does not compress or extend, and the ground motions are transmitted directly to the mass. Hence, the mass accelerations are identical to the base accelerations, including the maximums.

Similarly, at the low frequency end of the spectrum, the oscillator has a relatively low frequency. At this end, the spectral displacement converges to the maximum ground displacement. Once again, if the single DOF oscillator is considered, relative low frequency is characterized by a large mass relative to the spring stiffness. In this case, the large mass will tend to resist motion due to its inertia while the base moves up and down resisted only by a relative soft spring. From equation (2) for the relative coordinate $z(t)$, if the motion of the mass is zero, then the relative displacement $z(t)$ is identical to the ground displacement and as such the maximums

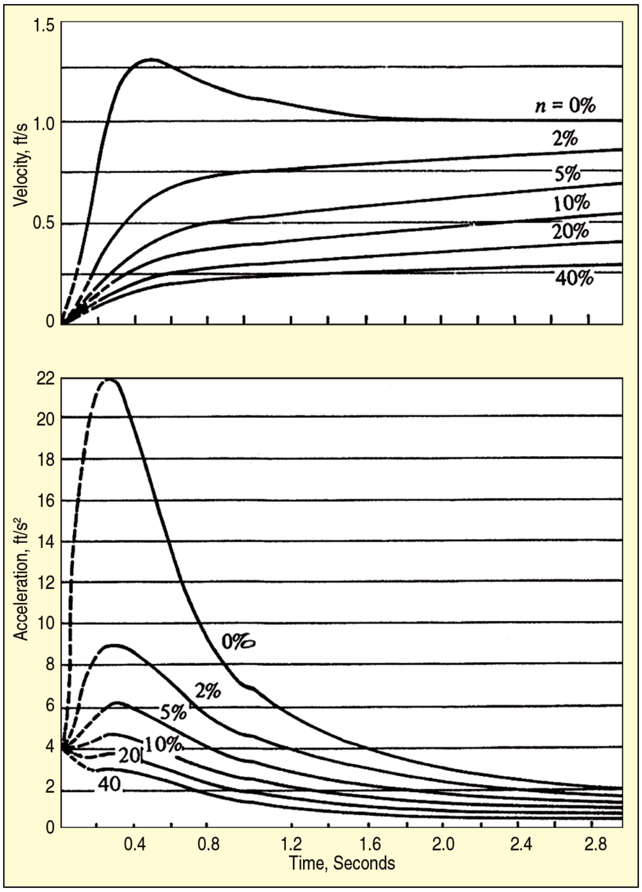

Figure 22. Housner design velocity and acceleration spectra.

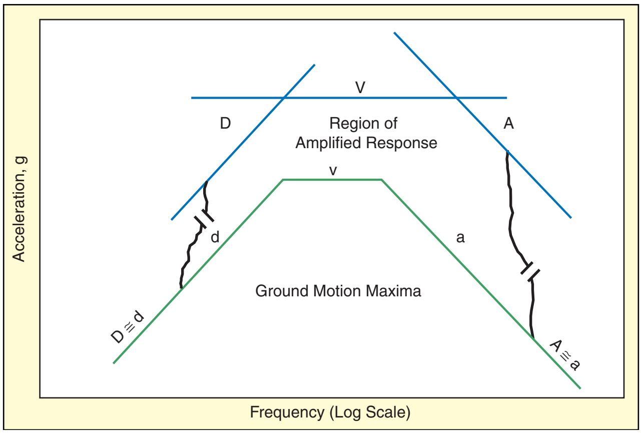

Figure 23. General shape of smoothed spectrum response.

are also identical.

Figure 24 illustrates one example of the construction of a design spectrum based on maximum ground motions. In this case the design spectrum is generated as a smoothed curve between the following three maximum conditions on acceleration, velocity

Table 1. Amplification factors for Newmark Spectrum

Spectral Quantity	Probability Level, %	Damping Ratio, % 0.5	2.0	5.0	10.0
S_D	50	1.97	1.68	1.40	1.15
	84.1	2.99	2.51	2.04	1.62
S_V	50	2.58	2.06	1.66	1.34
	84.1	3.81	2.98	2.32	1.81
S_A	50	3.67	2.76	2.11	1.65
	84.1	5.12	3.65	2.67	2.01

S_D = factor · d, S_V = factor · v, S_A = factor · a
d, v, $\underline{a}$ = maximum ground displacement, velocity, acceleration, respectively
$ad/v^2 \approx 6$

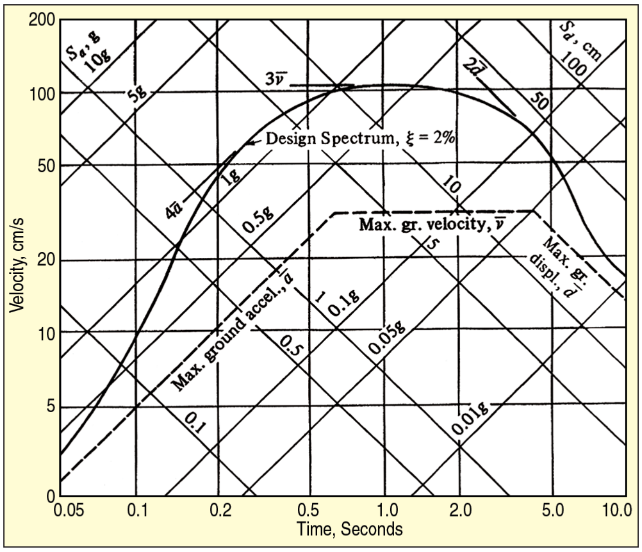

Figure 24. Design response constructed from maximum ground motion.

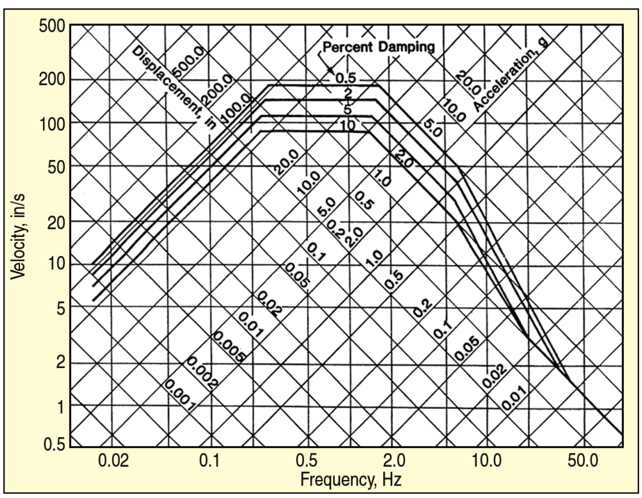

Figure 25. Newmark design spectra for alluvium soil (maximum ground acceleration = 1 g).

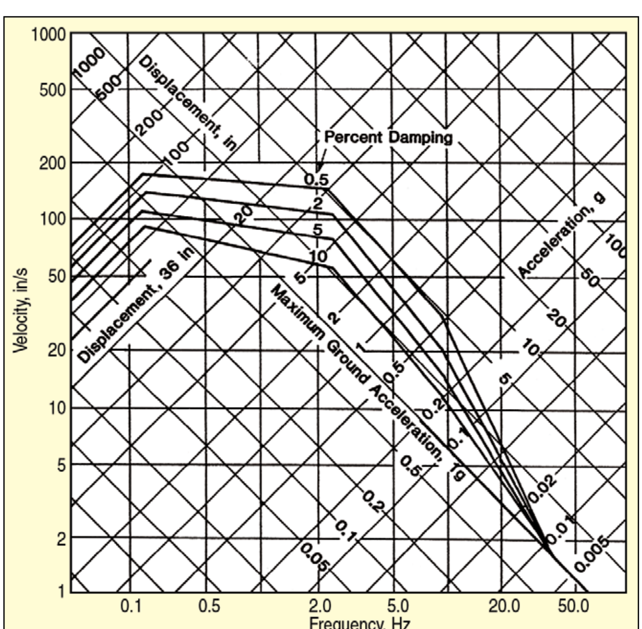

Figure 26. Atomic Energy Commission design spectra.

and displacement, respectively:

- Maximum SRS acceleration = four times maximum ground acceleration.
- Maximum SRS velocity = three times maximum ground velocity.
- Maximum SRS displacement = two times maximum ground displacement.

Figure 26 is an example of the development of a design shock response spectrum developed from the maximum ground motion that is quite similar to the Newmark Spectra of Figure 25. It was published in the 1970s by the Atomic Energy Commission for the seismic design of nuclear power plants.[7]

SRS Analysis of Linear MDOF Structures

Thus far the discussion of the application of a shock response spectrum has been only in the context of a single DOF system. Occasionally real equipment can be modeled adequately with only one degree of freedom, but more typically real systems will require a multi degrees of freedom model to adequately capture overall system dynamics accurately. When this is the case and the system must be designed to meet a prescribed shock response spectrum, a technique must be used to adapt the design SRS to a multi degrees of freedom structure. If the MDOF system can be modeled as a linear system, then the maximum dynamic system response can be approximated by the use of a technique called mode superposition. This technique transforms the system dynamics from physical coordinates into modal coordinates in such a way that each mode of the system behaves as if it were a single degree of freedom.

The shock response spectrum can then be applied to the system in the modal domain based on the mode frequency and the degree of participation that each mode has to the overall system response. The individual modal responses are then transformed back into physical coordinates using the modal transformation. The modal transformation between physical and modal coordinates is accomplished by an Eigen value extraction of the matrix equations of motion. The resulting Eigen values and Eigen vectors are the frequencies and mode shapes for of each mode of vibration of the system, respectively. The mode shapes, or Eigen vectors, are used to transform the system from physical coordinates to modal coordinates and vice versa. The mathematics to accomplish this transformation are summarized below.

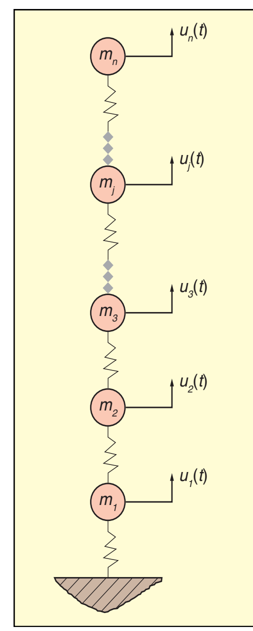

Figure 27. n degree of freedom nondampened linear system.

Consider the linear n-degree of freedom system shown in Figure 27. For this discussion, in the interest of keeping the mathematics relatively simple, damping is not included but the technique applies equally well to damped systems. As noted earlier, the system must be linear to use mode superposition. If the system contains nonlinear elements, the resulting equations of motion are nonlinear and it is not possible to do an Eigen value extraction of a system of nonlinear equations. This is a significant liability of mode superposition, since many real systems have significant nonlinearities.

The system in Figure 27 has n lumped masses with displacements $u_j(t)$. For this discussion, only vertical displacements are considered. However in general each mass could have six DOF; three translations and three rotations. As was done with the SDOF system from the first series of equations, relative coordinates are defined as the displacement of each mass relative to the displacement of the base. An equation of motion similar to (1) can be written for each of the n masses. Similar to (2), for the j^{th} mass a relative coordinate is defined as:

$$z_j(t) \equiv u_j(t) - u_b(t) \tag{10}$$

When the relative coordinate (10) is substituted into the equation of motion for the j^{th} mass and all n equations of motion are assembled into one matrix equation, the system equations of motion become:

$$[M]\{\ddot{z}(t)\}+[K]\{z(t)\}=-[M]\{1\}\ddot{u}_b(t) \qquad (11)$$

[M] and [K] are the mass and stiffness matrices of the system, respectively. Setting the right hand side equal to zero results in the free vibration equation:

$$[M]\{\ddot{z}(t)\}+[K]\{z(t)\}=\{0\} \qquad (12)$$

If simple harmonic motion of the form given by (13) is assumed:

$$\{z\}_i=\{\varphi\}_i \sin\omega_i t \qquad (13)$$

and substituted into (12), the Eigen value problem (14) results:

$$\omega_i^2[M]\{\varphi\}_i=[K]\{\varphi\}_i \qquad (14)$$

The Eigen values are given by ω_i^2 and the Eigen vectors are given by $\{\varphi\}_i$. The Eigen vector matrix is the assembled matrix of the Eigen vectors:

$$[\Phi]=\left[\{\varphi\}_1\{\varphi\}_2\{\varphi\}_3\cdots\{\varphi\}_n\right] \qquad (15)$$

The eigen vector matrix is used to transform relative physical coordinates $\{z(t)\}$ into modal coordinates $\{x(t)\}$:

$$\{z(t)\}=[\Phi]\{x(t)\} \qquad (16)$$

Substituting $[\Phi]\{x(t)\}$ into the equation of motion (11) and premultiplying by $[\Phi]^T$ yields:

$$[\Phi]^T[M][\Phi]\{\ddot{x}\}+[\Phi]^T[K][\Phi]\{x\}=-[\Phi]^T[M]\{1\}\ddot{u}_b \qquad (17)$$

The matrices $[\Phi]^t[M][\Phi]$ and $[\Phi]^t[K][\Phi]$ are diagonal matrices due to orthogonality properties of normal modes. Hence, the matrix equation (17) decouples into n-single DOF equations. The resulting equation for i^{th} mode of vibration is:

$$\ddot{x}_i(t)+\frac{\{\varphi\}_i^T[K]\{\varphi\}_i}{\{\varphi\}_i^T[M]\{\varphi\}_i}x_i(t)=-\frac{\{\varphi\}_i^T[M]\{1\}}{\{\varphi\}_i^T[M]\{\varphi\}_i}\ddot{u}_b(t) \qquad (18)$$

The coefficient of $x_i(t)$ in (18) is the squared circular frequency for the i^{th} mode:

$$\omega_i^2=\frac{\{\varphi\}_i^T[K]\{\varphi\}_i}{\{\varphi\}_i^T[M]\{\varphi\}_i} \qquad (19)$$

The coefficient of the base acceleration is defined as the mode participation factor (20). The mode participation factor determines the degree of "participation" the i^{th} mode has to the overall system response:

$$P_i=\frac{\{\varphi\}_i^T[M]\{1\}}{\{\varphi\}_i^T[M]\{\varphi\}_i} \qquad (20)$$

Substituting (19) and (20) into (18) yields:

$$\ddot{x}_i(t)+\omega_i^2x_i(t)=-P_i\ddot{u}_b(t) \qquad (21)$$

Other than for the participation factor, (21) is identical to the single DOF equation (3) used to obtain the spectral acceleration S_A in SDOF section. Differentiating (16) twice, the transformation back to the physical coordinate accelerations become:

$$\{\ddot{z}(t)\}=\sum_{i=1}^{n}\{\varphi\}_i\ddot{x}_i(t) \qquad (22)$$

After some mathematical manipulations, the absolute physical coordinate accelerations can be derived in terms of the modal coordinate accelerations given by:

$$\{\ddot{u}(t)\}=\{\varphi\}_1P_1\ddot{x}_1(t)+\{\varphi\}_2P_2\ddot{x}_2(t)+\{\varphi\}_3P_3\ddot{x}_3(t)+\cdots\{\varphi\}_nP_n\ddot{x}_n(t) \qquad (23)$$

If the modal coordinate accelerations are replaced with the absolute values of their maxima, (23) becomes (24). Notice that the equal sign has been replaced with a "less than or equal sign." This is because the maximum modal coordinate acceleration will not occur at the same time t:

$$\{\ddot{u}\}_{\max}\le\left|\{\varphi\}_1P_1|\ddot{x}_1(t)|_{\max}\right|+\left|\{\varphi\}_2P_2|\ddot{x}_2(t)|_{\max}\right|+\cdots+\left|\{\varphi\}_nP_n|\ddot{x}_n(t)|_{\max}\right| \qquad (24)$$

Recall from (7) that the absolute value of the maximum acceleration of the SDOF oscillator is the spectral acceleration. Hence, for the i^{th} mode of the system, the following holds:

$$|\ddot{x}_i(t)|_{\max}=S_{Ai} \qquad (25)$$

Substituting (25) into (24) yields:

$$\{\ddot{u}\}_{\max}\le\sum_{i=1}^{n}\left|\{\varphi\}_iP_iS_{Ai}\right| \qquad (26)$$

Since all of the modal maxima from (24) will not occur at the same time t, an estimate of max acceleration of the j^{th} mass is typically obtained using a root sum squared type approach given by (27):

$$\ddot{u}_{j\max}\cong\sqrt{\sum_{i=1}^{n}\left(\varphi_{ji}P_iS_{Ai}\right)^2} \qquad (27)$$

This development has demonstrated that using mode superposition in conjunction with the shock response spectrum, an estimate of the maximum dynamic response of a linear system can be approximated without solving the transient MDOF equations of motion.[11]

Naval Shock Design Spectra

History. Welch[8] was among the earliest to provide written guidance for naval shock structural design. In the 1940s shock design based on the "Static-G" method which did not account for differences in mounting/foundation frequency, the location of equipment in the ship or the ship type. The approach using envelope spectra to design shipboard equipment began about 1948.[9] The maximum envelope spectrum approach, as Biot discovered earlier, resulted in considerable over-design.

The so-called Shock Spectrum Dip was discovered in 1957 by serendipity. Strange results were observed from reed gages during full ship shock trials. The reeds corresponding to fixed base natural frequencies of equipment gave unexpectedly low results. The conclusion from this observation was that the equipment interacted with ship structure, thereby affecting the response spectrum at the base of the equipment. Other observations from ship shock trials showed that heavy equipment responded with lower shock levels than light equipment. This motivated experiments at the Naval Research Laboratory that explained the spectrum dip and the effect of modal mass on frequency response.

Shock Spectrum Dip. The Naval Research Laboratory performed experiments in the 1960s using the equipment in Figure 28 to demonstrate shock spectrum dip.[10,11] Three double cantilever beams each of approximately the same weight, but with different stiffness served as the test structures. Equipment mounted to non-rigid foundations fed back forces into the foundations that affected the motion in such a way to result in a dip in the shock spectrum at the fixed base natural frequency of the equipment as shown in Figure 29. It was observed that the spectrum values of major interest (at the equipment fixed base frequency) tend to lie in the region of valleys rather than in the vicinity of peaks of the shock spectrum. These experiments demonstrated that envelope spectra for design of equipment will lead to potential extreme over conservatism as shown between the differences of the maximum and minimum spectrum values of Figure 30. The data points from the test are plotted on the graph, and it is evident that the actual values lie near the minimum spectrum envelope.

Modal Mass Considerations. The U. S. Navy's method for shock qualification by analysis uses a procedure termed the Dynamic Design Analysis Method, commonly referred to as DDAM. This procedure defines design spectrum values for each mode of the system as a function of not only frequency, but also the modal mass of each mode. The modal mass for the i^{th} mode of the system is given in Eq. 28. The modal mass is determined by the Eigen vector for the mode of interest and the mass matrix of the system:

$$\bar{M}_i=\frac{\left[\{\varphi\}_i^T[M]\{1\}\right]^2}{\{\varphi\}_i^T[M]\{\varphi\}_i} \qquad (28)$$

The modal mass represents the effective mass contribution due to the i^{th} mode of the structure. Figures 31 and 32 show the results

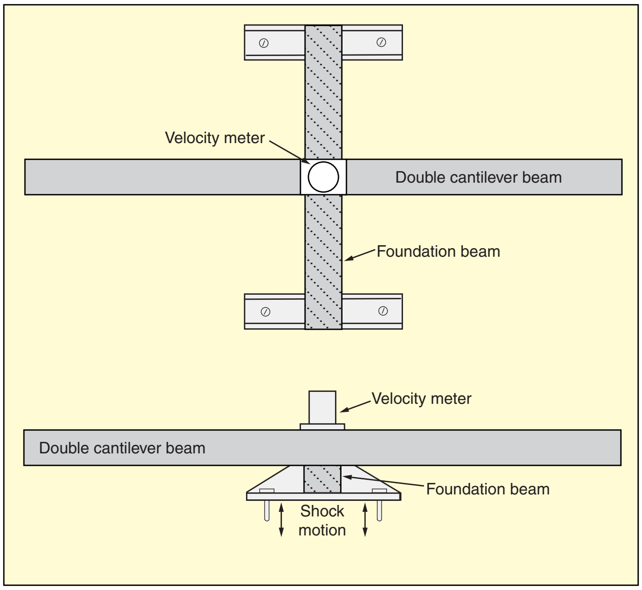

Figure 28. Naval Research laboratory experiment.[12]

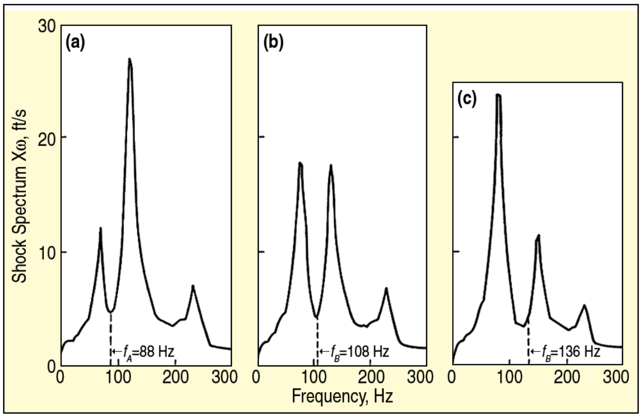

Figure 29. Shock spectrum for Beam A, Beam B, and Beam C.

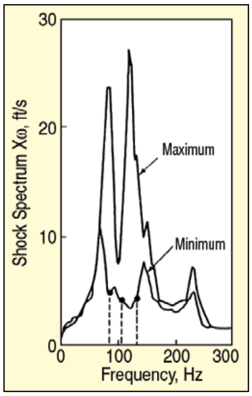

Figure 30. Maximum and minimum envelopes of all shock spectra.

from O'Hara and Cuniff study on modal mass.[13] From Figure 31, note that as the modal mass increases, the velocity and acceleration response decreases, which follows intuition.

The U. S. Navy uses the Dynamic Design Analysis Method for shock qualification of shipboard and submarine equipment. The DDAM method determines that the spectrum values to be used for each mode are based on the modal mass (28) and frequency. Figure 32 shows different design spectra based on the modal mass. Modes with greater modal mass are subject to lower maximum spectrum values and vice versa. DDAM also takes the aforementioned spectrum dip into consideration for determining the maximum spectrum values to be used for each mode.

Summary

An overview of the shock response spectrum has been presented primarily with the intent of informing the reader about its salient features, the history behind its development, and typical ways it is applied to the design of structures.

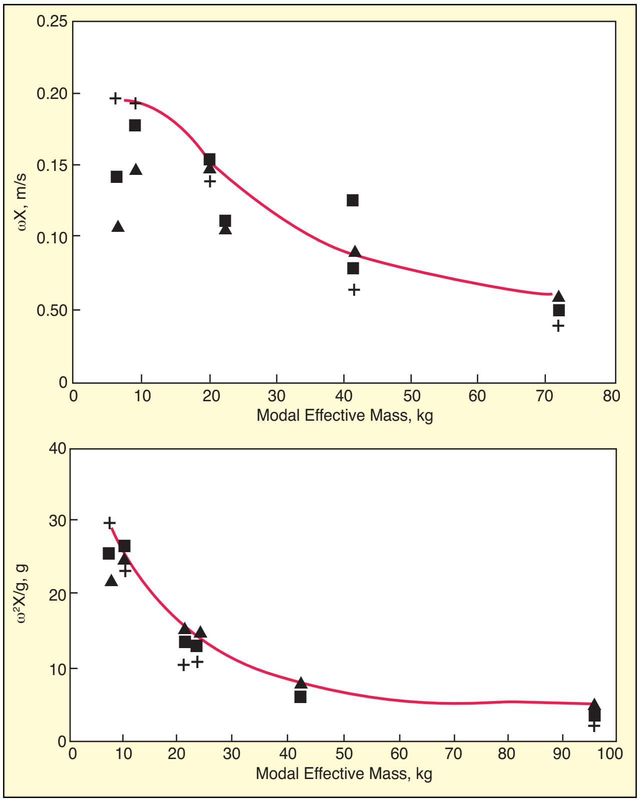

Figure 31. Velocity and acceleration as function of modal mass.

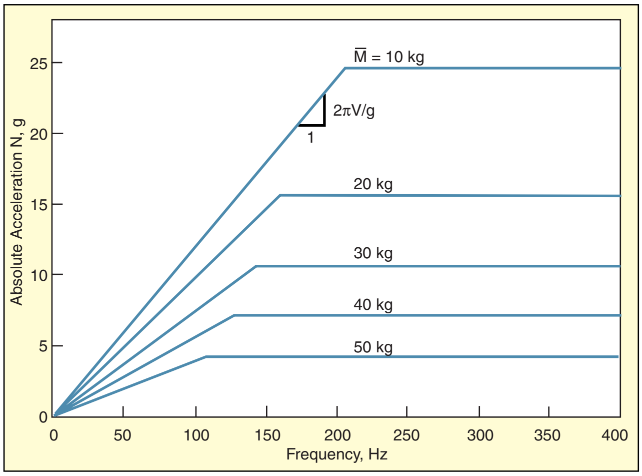

Figure 32. Shock spectra as a function of fixed base frequencies and modal mass.

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