The Vincent Circle

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More than 30 years ago, A. H. Vincent of Westland Helicopters demonstrated that if a structure is excited harmonically at a fixed frequency, the response at another position will trace a circle in the complex plane as a result of a stiffness modification between two points. This can be more generally expressed in the following manner. The structural or acoustical response at any position will map a circle in the complex plane for any straight line impedance modification in the complex plane. This article reviews the basis for this little-known principle and illustrates its usefulness for vibro-acoustics problems. The principle is demonstrated for noise radiation from a cantilevered plate and operator equipment cab. The technique can also be applied to waveguides.

Noise can sometimes be mitigated by making a change to a structural energy path by adjusting mass, stiffness, damping, or some combination of the three. Certainly, numerical simulation can often be used to guide the design process. However, we believe that an over-reliance on numerical simulation can mask important design insights that may be acquired using basic engineering principles. We believe that the principle discussed in this article is a useful observation that can aid engineers in making improvements to products.

Discovered by A. H. Vincent[1] of Westland Helicopters in 1972, it was largely overlooked in the intervening years until a recent paper by Tehrani, *et al.*[2] Vincent[1] noted that introducing a stiffness modification and varying the stiffness from minus to plus infinity would result in a circle when the displacement response was plotted in the complex plane. Vincent limited his scope to structures excited at a single point with a one-dimensional spring added between two positions on a structure. More than 30 years later, Tehrani[2] discovered that the principle could be generalized to an impedance modification in one dimension, incorporating mass and damping modifications.

Recall that mechanical impedance Z_M can be expressed as

$$Z_m = c_D + j\left(-\frac{k}{w} + \omega m\right) \qquad (1)$$

where k, m and c_D are stiffness, mass and damping respectively. According to the principle, changing c_D or the quantity $-k/\omega + \omega m$ (i.e., the real and imaginary part of Z_M respectively) from minus to plus infinity will trace a circle in the complex plane. This impedance modification is at one position and in a single coordinate direction. The minimum value of the response corresponds to the point on the circle closest to the origin of the complex plane.

Using the principle, the entire response region can be described for a structural modification introduced between two points or between a single point and ground. Use of Vincent's circle had previously been limited to vibration suppression problems and has been proven valid for structural loads, dynamic stiffness modifications and vibratory responses. However, our recent work[3] demonstrated that the principle can be generalized for both mechanical and acoustic impedance modifications and acoustic responses.

A Simple Example

The principle is illustrated via a plate with a single-force excitation. Figure 1 shows a plate cantilevered on one edge excited by a point force. A spring-mass-damper modification was introduced at point *r*. Numerical simulation was used to calculate the sound pressure (in air) at point *q* due to a force applied at point *p*. All simulations were performed at a frequency of 140 Hz.

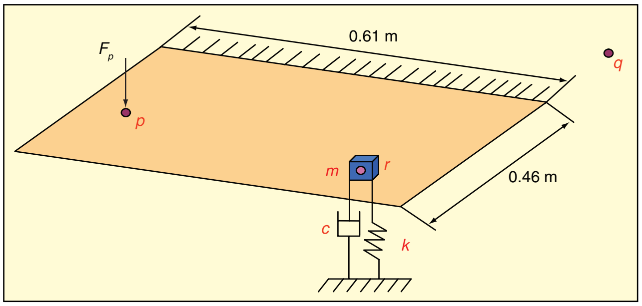

Figure 1. Schematic of cantilevered plate, approximate location of mechanical impedance modification at point r, and point q for acoustic response.

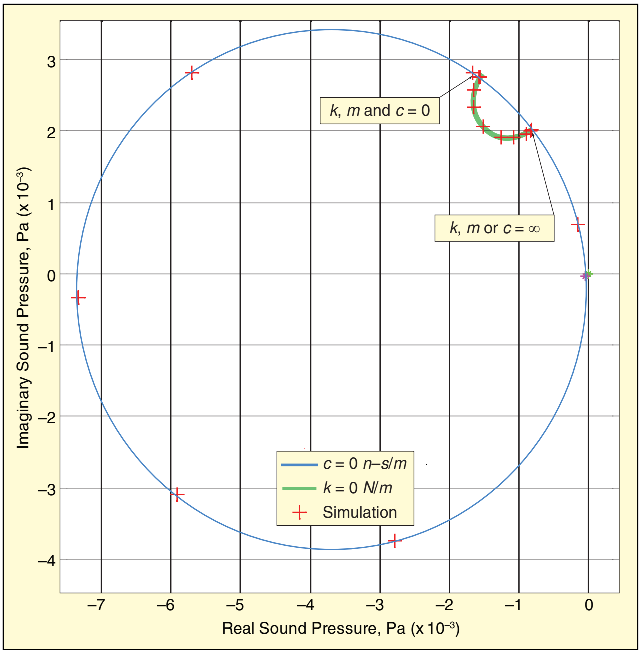

Figure 2. Sound pressure at point q (for a unit force at point p) plotted in a complex plane as a function of real and imaginary modifications to mechanical impedance.

The results are shown for two modifications in Figure 2. The large circle is for a stiffness *k* and/or mass *m* modification with the damping c_D set to zero. This corresponds to varying the imaginary part of the mechanical impedance. Similarly, the smaller arc is for a damping modification (real mechanical impedance modification) with the stiffness and mass set to zero. Note that the circle and arc intersect at two points. One point occurs when all three constants (k, m, and c_D) are zero (unmodified case). The other is when either k, m, or c_D is infinite. The smaller circle divides the larger circle into two separate arc lengths. The longer and shorter arc lengths correspond to stiffness and mass modifications respectively with the other set to zero. An optimum modification can be identified as that point where the response is a minimum (closest to the origin of the complex plane).

In Figure 3, the damping is plotted on the vertical axis and a series of circles are traced in the different complex planes as the real part of the mechanical impedance (damping) is increased. The circles have a smaller diameter for higher values of damping, since damping tends to reduce the differences between the amplitude of vibration at resonances and anti-resonances. As expected, the circles move toward the origin of the complex plane as the damping is increased.

Application to Mechanical Impedance

The development of the method below is similar to that shown by Done and Hughes.[4] Figure 4 shows a schematic of a structure with a modification between points *r* and *s*. The structure is excited at point *p*, and the response will be computed at point *q*. Done and Hughes supposed that point *q* was on the structure, and the response was a structural vibration. However, the derivation here

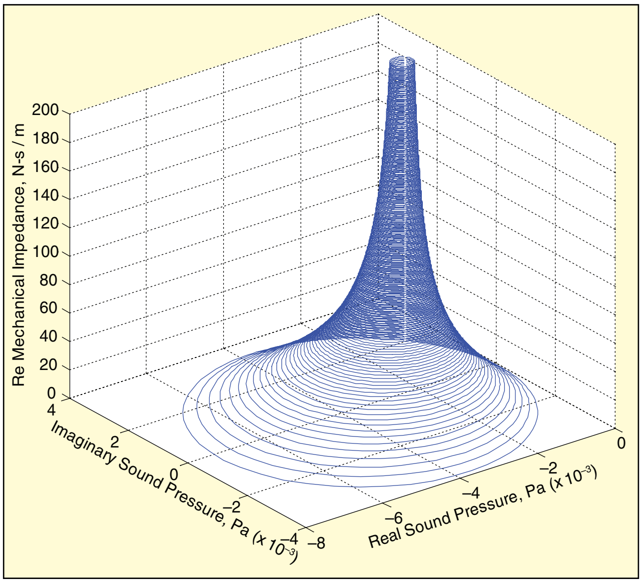

Figure 3. Sound pressure at point q (for a unit force at point p) plotted as a function of real and imaginary modifications to mechanical impedance. Real part of mechanical impedance is increased along the vertical axis.

assumes a structural force at point p and an acoustic response at a point q.

Assume that the spring is replaced by two forces F_r and F_s. In this case, the vibration responses at points r and s in the direction of the mechanical impedance and the acoustic response at point q can be written in terms of the applied forces F_p, F_r, and F_s. Thus:

$$p_q = H_{qp}F_p + H_{qr}F_r + H_{qs}F_s \tag{2a}$$

$$v_s = H_{sp}F_p + H_{sr}F_r + H_{ss}F_s \tag{2b}$$

$$v_s = H_{sp}F_p + H_{sr}F_r + H_{ss}F_s \tag{2c}$$

where H_{ij} are the unmodified transfer functions (determined prior to any impedance modification) between the vibration (velocity) or acoustic responses at point i and the forces or acoustical inputs at point j.

Note that the forces F_r and F_s can be expressed in terms of the mechanical impedance, Z_M, and the velocity responses, v_r and v_s, as:

$$F_r = Z_M\left(v_S - v_r\right) = -F_s \tag{3}$$

and then substituted into Equation 2. This results in a set of three simultaneous equations with three unknown responses, x_r, x_s, and p_q. Solving for the modified transfer function (p_q/F_p), the following expression is obtained:

$$\frac{p_q}{F_p} = H_{qp} + \frac{Z_m\left(H_{sp} - H_{rp}\right)\left(H_{qr} - H_{qs}\right)}{1 = Z_M\left(H_{rr} + H_{ss} - H_{rs} - H_{sr}\right)} \tag{4}$$

Note that Equation 4 is of the form:

$$\frac{p_q}{F_p} = b + \frac{Z_M c}{1 + Z_M d} \tag{5}$$

where b, c, and d are complex numbers defined as:

$$b = H_{qp} \tag{6a}$$

$$c = \left(H_{sp} - H_{rp}\right)\left(H_{qr} - H_{qs}\right) \tag{6b}$$

$$d = \left(H_{rr} + H_{ss} - H_{rs} - H_{sr}\right) \tag{6c}$$

Tehrani[2] noted that Equation 5 is a form of the Moebius transformation, which can be expressed as:

$$Z = \frac{\alpha z + \beta}{\gamma z + \delta} \tag{7}$$

where α, β, γ, and δ are complex numbers. Equation 5 can be converted to the form of Equation 7 by just putting the right-hand side under a common denominator. The Moebius transformation

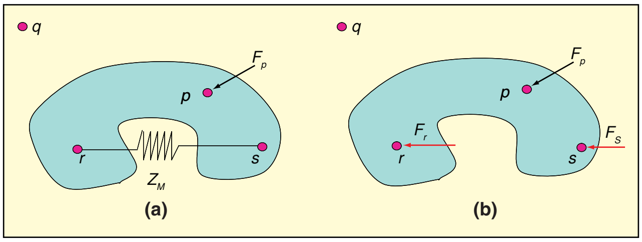

Figure 4. (a) Excited structure and location of modification; (b) mechanical impedance replaced by forces.

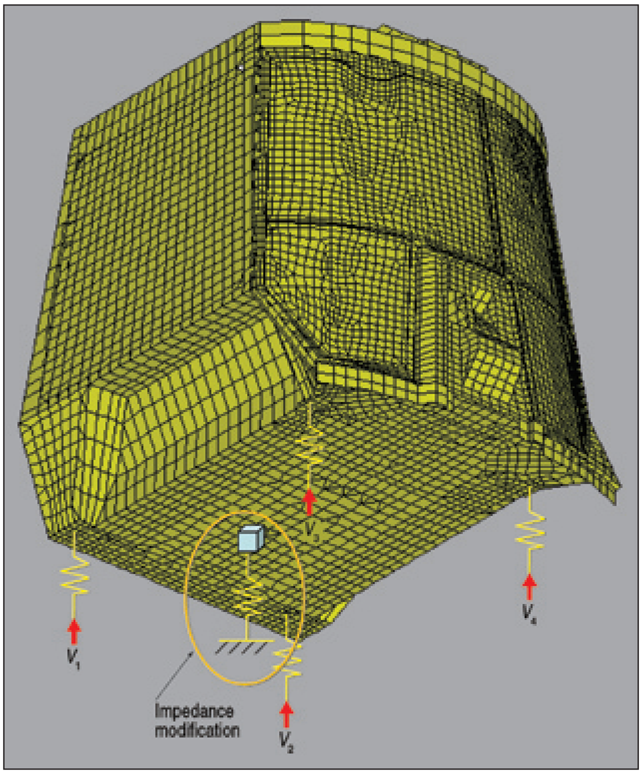

Figure. 5 Analysis model of operator equipment cab showing the impedance modification; analyses conducted using ANSYS and SYSNOISE.

states that a straight line or circular modification of z in the complex plane will result in a straight line or circle of Z when traced in the complex plane.[5]

Also note that the transfer functions H_{ij} need not be found *a priori*. Instead, the complex constants b, c and d can be solved by making three known impedance (Z_M) modifications to an analysis model or a real structure and measuring the response at a point. Once the complex constants b, c and d are determined, the response to any impedance (Z_M) modification can be determined.

A more detailed development is presented in Reference 3, where we demonstrate that the principle can be applied for:

- Multiple input forces
- Both vibratory and acoustic responses
- Both vibratory and acoustic impedance modifications
- Both series and parallel impedances

For the sake of brevity, we will focus on a few applications in this article. (See Reference 3 for a more rigorous explanation.)

As an aside, note that modifying the frequency in Equation 1 results in a straight line modification of mechanical impedance in the complex plane. This equation is the system impedance for a single degree of freedom mass, spring and damper combination. For viscous damping (proportional to velocity), the mobility will trace a circle in the complex plane as frequency is varied for a single-degree-of-freedom system.[6] Indeed, the so-called *modal circle*[6] is a special case of the Moebius transformation.

Operator Equipment Cab with Multiple Inputs

Finite- and boundary-element analyses of an operator equipment cab were conducted using ANSYS and LMS SYSNOISE

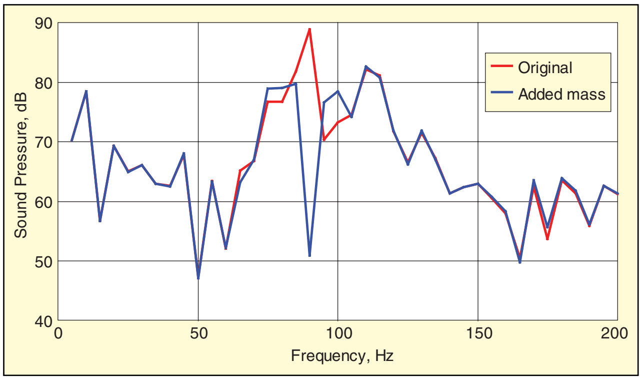

Figure 6. Comparison of sound pressure at driver's ear in operator equipment cab with and without impedance modification.

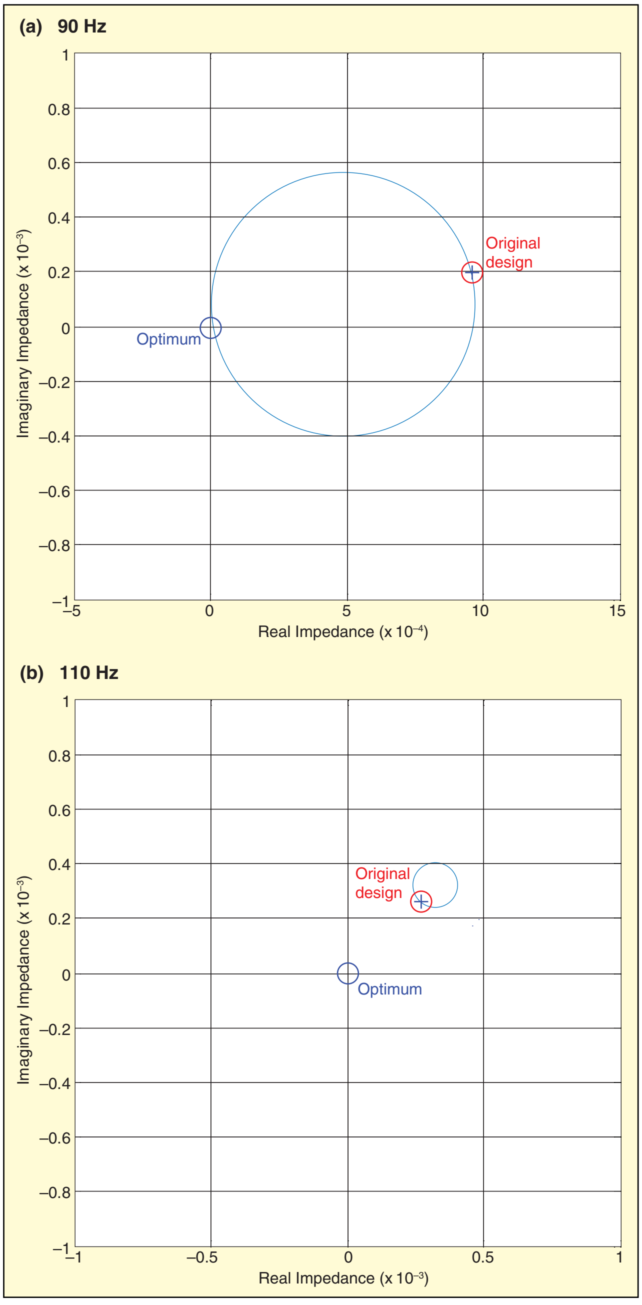

Figure 7. Acoustic response plotted in complex plane at 90 and 110 Hz for imaginary impedance (mass and/or stiffness) modification.

respectively, and the results demonstrate how the Moebius transformation might be applied. Figure 5 shows the finite-element model of the operator equipment cab. The four input vibrations at the four mounts and a mechanical impedance modification to the floor are indicated. The finite-element model for the cab was used to predict the structural vibration by a forced-response analysis. The computed structural vibration was used as the velocity boundary condition for a subsequent boundary element analysis. The sound pressure response was computed at the location of the driver's ear.

The sound pressure at the driver's ear for the original unmodified design is shown in Figure 6, where peaks may be seen at 90 and 110 Hz. Our objective was to add a modification to the cab that would suppress the sound pressure at 90 Hz while not adversely increasing the sound pressure at 110 Hz. An imaginary impedance modification was considered (i.e., a modification to the mass and/or stiffness). Computed Vincent Circles in the complex plane are shown in Figure 7 for 90 and 110 Hz. Notice that the 90-Hz circle indicates that adding a modification at the selected position can suppress the noise. Conversely, the 110-Hz circle indicates an impedance modification will not have a significant effect on the response at 110 Hz.

Figure 6 shows the effect of adding a mass at the point shown in Figure 5. Notice that the sound pressure response is attenuated at 90 Hz, while there is little effect at 110 Hz. The added mass shifts an anti-resonance to 90 Hz. The results demonstrate how the principle can be used to assess the location and effect of impedance modifications. In principle, the approach can be applied experimentally as well.

Application to Mufflers and Silencers

The principle is also applicable to acoustic impedance. Reference 3 details how the Vincent Circle is also applicable to acoustic impedances in waveguides. A muffler or silencer system can be thought of as a waveguide in which the sound propagates. At lower frequencies, the duct cross-sectional dimensions are small compared to the acoustic wavelength. Accordingly, assume that plane waves propagate inside the duct system, thereby simplifying the analysis. In this case, a duct system can be described as an acoustic network and sound propagation in the network is simulated using the transfer matrix method popularized by Munjal.[7]

For example, Figure 8 shows an exhaust system with a variety of impedance elements. The approach is valid for both series and parallel impedances. Series impedances include the source (Z_s), termination (Z_T), and transfer (Z_{tr}) impedances. The source (Z_s) and termination (Z_T) impedances dictate how much sound is reflected back from the source and termination, respectively, while transfer impedances (Z_{tr}) are often used to model perforated elements. Parallel or branch impedances (Z_B) are used to simulate quarter-wave tubes and Helmholtz resonators.

An application is shown in Figure 9. Our objective was to modify the impedances Z_1 and Z_2 by changing the lengths of the ducts to attenuate noise at both 120 and 300 Hz. Figure 10 shows the resulting transmission loss. Though it is straightforward to add a quarter-wave tube or Helmholtz resonator to an exhaust system, the configuration shown in Figure 9 is more difficult to optimize because impedances Z_1 and Z_2 are in parallel with each other. However, the Vincent Circle aids in selecting appropriate lengths.

Conclusions

We have demonstrated that the Vincent Circle aids in understanding the impact of mechanical and acoustic impedance modifications on a vibroacoustic system. The Principle is applicable to both series and parallel impedance modifications.

It is limited to single-degree-of-freedom, lumped-element modifications. Certainly, this is a weakness. For example, a modification like changing the thickness of a panel could not be directly considered by means of the Vincent Circle as developed in this article. In the case of duct acoustics, changing the length of an expansion chamber modifies the impedance at two locations, and the principle is not directly applicable.

Similarly, application of the Moebius transformation is limited to analysis at a single frequency. It is not as useful if the structural or acoustic excitations are not tonal in nature, since moving resonant frequencies will be of little benefit. That being the case, the method is most applicable when anti-resonances can be assigned

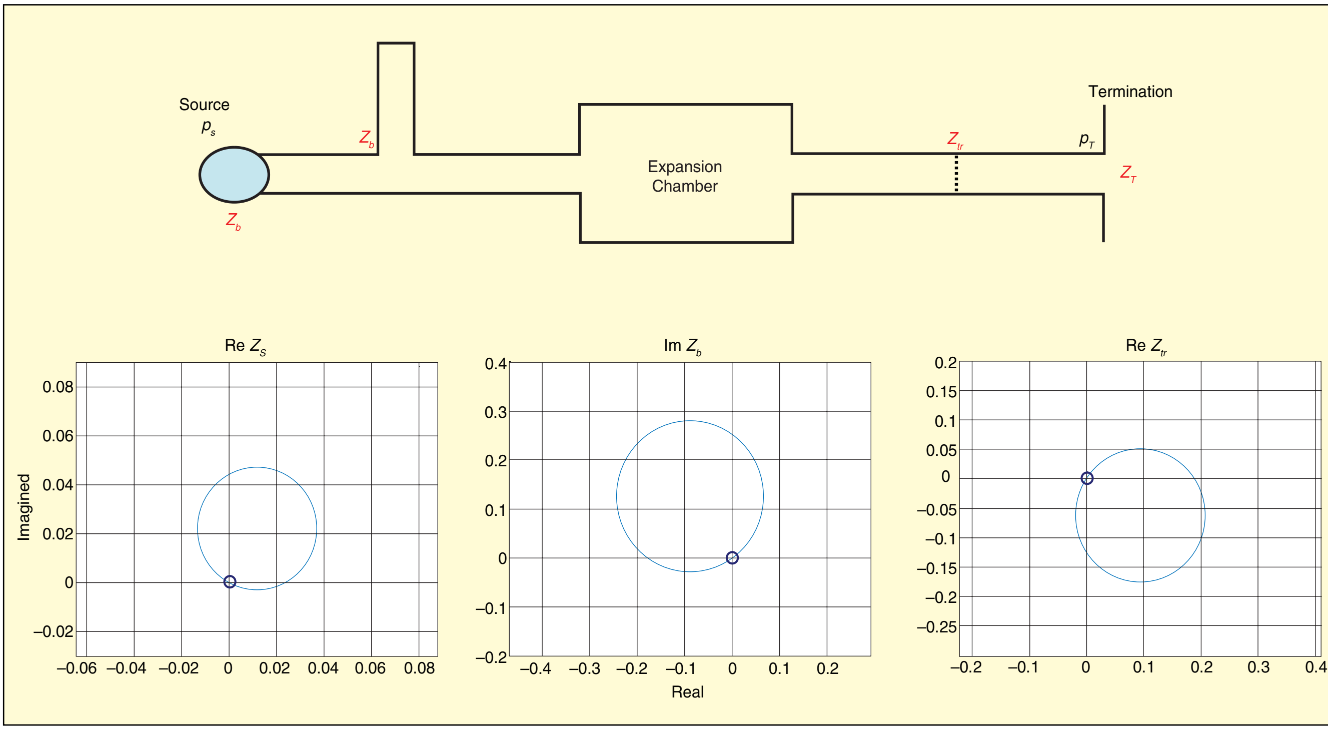

Figure 8. Application of the Vincent circle to a waveguide. Vincent circles are shown for real (Z_S and Z_{tr}) and imaginary (Z_b) impedance modifications.

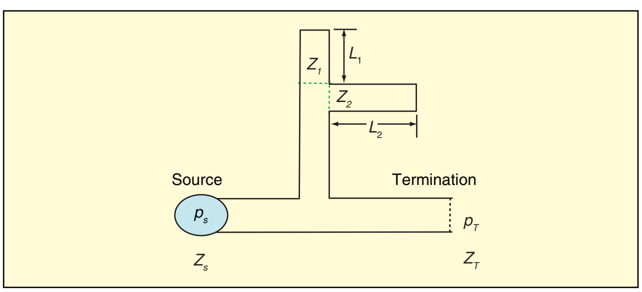

Figure 9. Schematic showing waveguide with side branch including acoustical impedances (Z_1 and Z_2) in parallel.

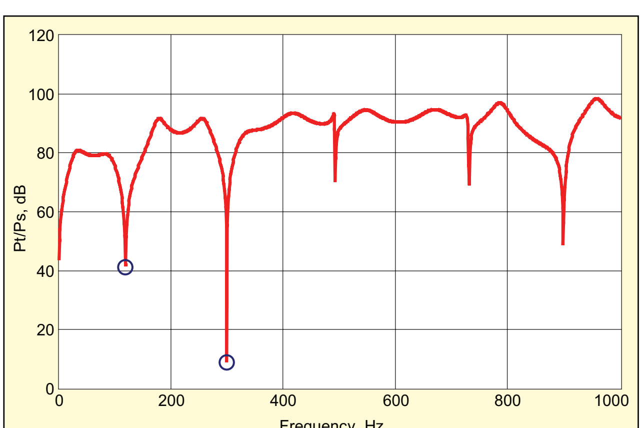

Figure 10. Transfer function between termination and source sound pressures after lengths (acoustic impedances) for waveguide in Figure 9 were optimized.

to important excitation frequencies. Fortunately, excitations are generally tonal for compressors, internal combustion engines, and many other sources of noise and vibration.

One drawback to the approach is that single-degree-of-freedom, lumped-element modifications are difficult to implement in practice, especially in the case of mechanical impedance. For example, the addition of a translational spring or damper generally also modifies the mechanical impedance in the other translational and rotational directions. Even if the impedance modification is precisely controlled, the local damping will likely be changed.

Nonetheless, the approach can be applied experimentally in principle. In fact, Tehrani, *et al.*[2] successfully utilized the Vincent Circle to minimize the structural vibration on a beam. Anti-resonances were identified using the Vincent Circle, and masses were added at selected locations. In practice, mechanical impedance is most easily controlled by changing the mass. If mass is welded to a structure, the change in damping (real part of the impedance) should be minimal compared to the change in mass (imaginary part of the impedance).

Though it may not be practicable to assign impedance precisely, understanding the influence of impedance modifications has intrinsic value. For instance, by varying the impedance of a lumped element located on a panel, the impact of adding or subtracting mass, stiffness, or damping at a particular location can be assessed.

Acoustic impedance modifications in ducts are more easily controlled than mechanical impedances. As previously noted, the method is amenable to any branch or series impedance modification. For the case of a side branch, the impedance can most easily be adjusted by changing the length. Likewise, the impedance of a Helmholtz resonator can be tuned by the equivalent length, resonator volume, and side branch area.[7]

Acknowledgments

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