Random Vibration – A Brief History

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Humans have observed and experienced dynamic random phenomena for millennia through our contact with earthquakes, winds, ocean waves, rough roads and trails. Before people could easily conceptualize harmonic motions, they observed random vibration. Today, random vibration is thought of as the random motion of a structure excited by a random input. The mathematical theory of random vibration is essential to the realistic modeling of structural dynamic systems. This article summarizes the work of some key contributors to the theory of random vibration from its inception in 1905, with the work of Einstein, to the present. Several graphical examples are included.

In 1827, upon observing the motion of particles of pollen in a fluid suspension, Robert Brown, a Scottish botanist, speculated that particle motions were due, not to some vitality in the particles, but to molecular-kinetic motion in the fluid. That is, he speculated that unobservable particles in the fluid were impacting the particles from the pollen to excite their motion. The motion became known as Brownian motion.

In 1905, Albert Einstein wrote the first paper on random vibration,[1,2] "On the Movement of Small Particles Suspended in a Stationary Liquid Demanded by the Molecular-Kinetic Theory of Heat." (Einstein wrote several other famous papers in 1905, among them, his paper on the special theory of relativity and his paper on the photoelectric effect. He won the Nobel Prize in Physics for the latter work.) He developed equations governing the distribution of motions of a particle suspended in a fluid in a derivation understandable to most undergraduate students of thermodynamics. Along the way, he developed a way of understanding random processes because the theory of random processes did not exist at that date, at least, in the form it exists today. Einstein's work spawned a flurry of activity in random vibration.

This article summarizes the work of Einstein and some of those who followed in his footsteps. It summarizes the milestones in random vibration from 1905 to the present, including development of an alternate to Einstein's technique for analysis of random vibration, definition of the spectral density of a stationary random process, development of the fundamental relation of random vibration in scalar and matrix forms, estimation of spectral density, specification of nonstationary random processes, random vibration of random structures, and many others. Many examples are provided.

Einstein's Introduction of Random Vibration

By the start of the twentieth century the idea that gases and fluids might be composed of molecules that move freely and energetically was well established. In fact, Robert Brown had speculated as early as 1827 that that the motion of inert particles in a fluid medium is caused by the molecular-kinetic effect, the impingement of unobservable particles in a fluid medium upon a microscopic, observable, particle. If we were to observe the Brownian motion of a particle suspended in a fluid, we would see a sequence like that shown in Figure 1.

Through 1905 a mathematical theory for the so-called Brownian movement had not been developed. In 1905 Albert Einstein[1,2] developed a mathematical theory to describe Brownian movement. Einstein did not use a direct approach to solve the problem. (A direct approach would write the equation governing motion of the particle in Brownian motion, and then find the probabilistic response character from the character of the input.) Rather, he argued that if the molecular-kinetic theory of heat is applicable in describing the Brownian motion of particles, then pressures on Brownian particles can be established. Those pressures would cause the diffusion of small particles in a suspension. (He assumed the particles to be spherical.) He developed a diffusion equation governing the probability density function (PDF) of a collection of Brownian
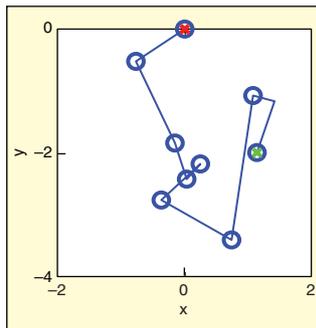

Figure 1. The locations of a Brownian particle on the x-y plane, observed at a fixed interval of time, and on a pre-established length scale. The particle starts at the origin (red circle), and ends at the tenth observation (green circle).

particles. During development of the diffusion equation, he made assumptions that would later characterize the molecular excitation as an ideal white noise, an excitation with signal content over a very broad band of frequencies. (See the following section for description of white noise.) The diffusion equation Einstein obtained is (for a single, one-dimensional component of motion):

$$\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2} \quad f(x,0) = \delta(x) \quad (1)$$

where $f(x,t)$, $-\infty < x < \infty$, $t \geq 0$ is the PDF of particle displacement at displacement location x and at time t and D is the coefficient of diffusion. The initial condition at $t = 0$ specifies that the PDF starts as a Dirac delta function; that is, there is complete certainty that displacement motion starts at the origin. For Einstein's formulation the coefficient of diffusion is $D = (RT/N)(1/6\pi c_f r)$ where R is the universal gas constant, T is the absolute temperature, N is Avagadro's number, c_f is the coefficient of viscosity of the fluid, and r is the radius of the sphere.

The solution to Eq. 1 is:

$$f_{X(t)}(x) = \frac{1}{\sqrt{(2\pi)(2Dt)}} \exp\left(-\frac{x^2}{2(2Dt)}\right) \quad -\infty < x < \infty, t \geq 0 \quad (2a)$$

where the subscript on f indicates that we are interested in the displacement random variable, $X(t)$, at time t. Note that the standard deviation of the response is:

$$\sigma_X(t) = \sqrt{2Dt} \quad t \geq 0 \quad (2b)$$

The standard deviation of the displacement response grows, without bound, as the square root of time. The reason is that there is no force applied to the system (a spring) that causes the displacements to oscillate about the origin. If we were interested in characterizing the response of a Brownian particle we would use Eq. 2 to answer questions regarding the unfolding of random particle response in time.

However, structural dynamicists are interested in considering structural response. The simple mechanical system equivalent to the one Einstein considered is the structure shown in Figure 2; it is a mass tied to ground with a damper (and no spring). Let m and c be the mass and damping constant of the structure with units of lb-sec^2/in and lb-sec/in, and let S_{ww}, be the two-sided spectral density of the white noise excitation with units of lb^2/(rad/sec). Equivalence between the mechanical system and the Brownian particle is established when $D = S_{ww}m^2/2c^2$. Figure 3 shows five marginal PDFs of the displacement response at normalized times $\tau = 2Dt$: 0.1,1,4,7,10 and the minus/plus one standard deviation curves. The mechanical system parameters are arbitrary and the response depends only on the coefficient of diffusion D.

Einstein did not consider explicitly the time-domain response of the system of interest; however, we can do so, easily. The equation governing motion of the system in Figure 2 is:

$$m\ddot{x} + c\dot{x} = w(t) \quad x(0) = x_0, \dot{x}(0) = v_0, t \geq 0 \quad (3)$$

where $x(t)$, $t \geq 0$ is the displacement response, dots denote differentiation with respect to time, the initial conditions specify the response, and $w(t)$, $t \geq 0$ is a white noise realization from a random source. (The response $x(t)$ is not capitalized, here, be-
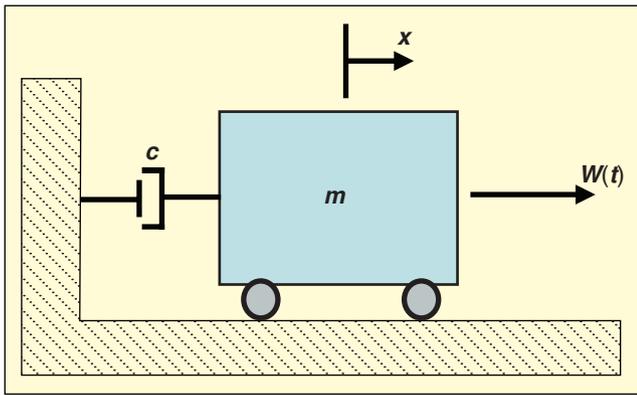
Figure 2. The structural dynamic system equivalent to the system analyzed by Einstein.
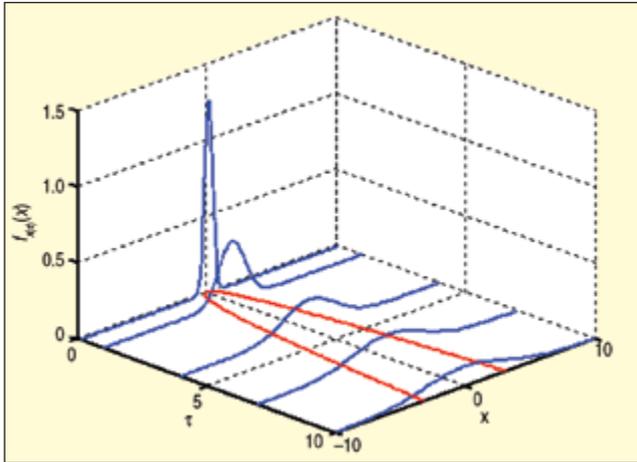

Figure 3. Five separate marginal PDFs of the displacement response of the system in Figure 2 at normalized times $\tau = 2Dt = 0.1, 1, 4, 7, 10$. (Blue, Eq. 2a, plotted above the $\tau - x$ plane) Standard deviation of the response. (Red, Eq. 2b, plotted in the $\tau - x$ plane).
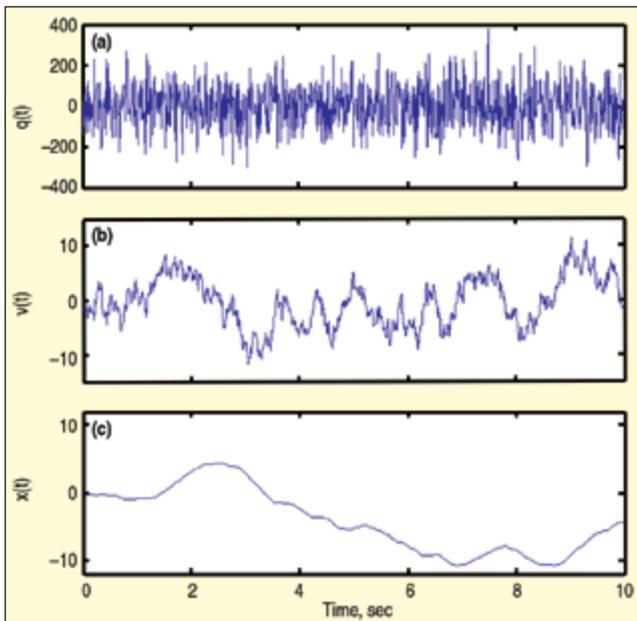

Figure 4. (a) Sample excitation from a band-limited random source with two-sided spectral density, $S_{ww} = 1$ $lb^2/(rad/sec)$, with frequency content in [–50,50) Hz; (b) velocity response of structure of Figure 2, with $m = 1$ lb-sec²/in, and $c = 1$ lb-sec/in, to the input in Figure 4a. (c) displacement response of the structure of Figure 2, with $m = 1$ lb-sec²/in, and $c = 1$ lb-sec/in, to the input in Figure 4a.

cause we consider the exciting force to be a sample of the input random process, not the random process, itself.) We can generate a sample excitation[3] from the white noise random process, and then compute the velocity and displacement responses the input
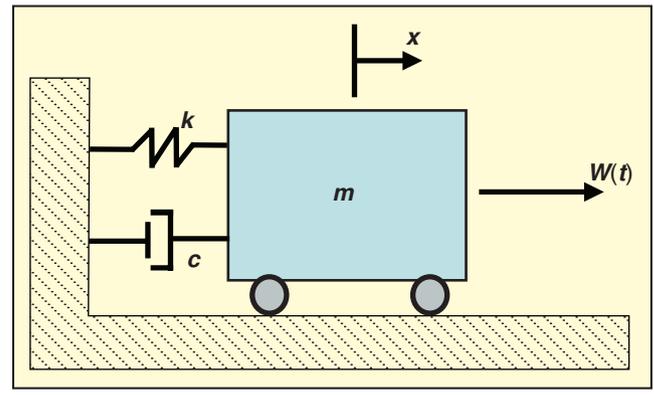

Figure 5. Single-degree-of-freedom (SDOF) structure with mass m stiffness k and damping c excited by input force W(t).

excites. (The excitation must approximate the ideal white noise with band-limited white noise.) Figure 4a shows a realization of band-limited white noise excitation with signal content in the frequency band [–50,50] Hz and two-sided spectral density S_{ww} = 1 lb^2 / (rad/sec). Figure 4b is the velocity response of a system with mass, $m = 1$ lb-sec²/in, and damper, $c = 1$ lb-sec/in. Figure 4c is the displacement response of the same system. Note that the input is simply one excitation from an infinite ensemble of possibilities; therefore, additional trials would yield other inputs and corresponding responses.

Because the problem Einstein solved yields the probabilistic description of the motion of a mass attached via a viscous damper to a fixed boundary and excited with white noise, his development can be thought of as the first solution to a random vibration problem and the dawning of the era of random vibration analysis.

However, the fact is that structural dynamicists are more interested in structures that are supported on resilient elements; such structures usually display oscillatory responses. The simplest form of oscillatory structure is the single-degree-of-freedom (SDOF) structure shown in Figure 5. This is the fundamental model of structural dynamics, one whose response is treated in practically every text on random vibration. (See, for example, the fundamental texts,[4,5] the intermediate texts,[3,6] or the more advanced texts.[7,8]) The first successful treatments of such a structure were by Smoluchowski[9] and Furth[10], independently in 1916 and 1917. They developed a form of the diffusion equation for the SDOF structure that would, eventually, become known as the Fokker-Planck equation. Interestingly, the case they considered is the over-damped (non-oscillatory) system. They referred to the system we call the SDOF structure as the "harmonically bound particle."

The limitation of over-damping would eventually be overcome, but in the short term, Ornstein[11] developed the idea of analyzing random vibrations, directly, based on the governing equation of motion. The equation governing motion of the SDOF structure is

$$m\ddot{X} + c\dot{X} + kX = W \qquad (4)$$

where m, c, X and W are defined as before, and k denotes the stiffness of the spring that attaches the SDOF mass to ground. (The excitation and response, $W(t)$ and $X(t)$ are capitalized, here because they are formally considered to be random processes.) The initial conditions on this equation must involve the joint probability distribution of displacement and velocity. The method Ornstein developed forms the basis for the method we use today to perform random vibration analysis.

Uhlenbeck and Ornstein[12] wrote a paper that elaborated on Ornstein's solution to Eq. 4. In that paper they described the excitation and response in terms of their moments (averages). This use of moments was an extremely important innovation; to a great degree, it constructed a foundation for the framework most random vibration analysts use today for the solution of random vibration problems. Specifically, to start, they described the input excitation as having zero mean and autocorrelation function (a term not in use at the time) defined:

$$E\big[W(t_1)W(t_2)\big] = R_{ww}(t_1, t_2) = 2\pi S_{ww}\delta(t_1 - t_2) \qquad -\infty < t_1, t_2 < \infty \quad (5)$$
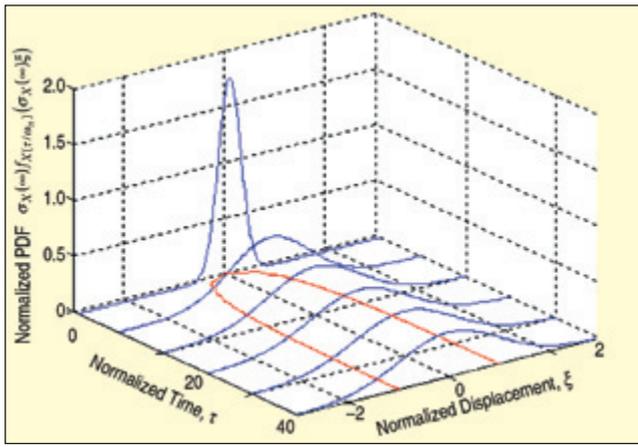
Figure 6. Six marginal PDFs of the displacement response of an SDOF structure developed by Uhlenbeck and Ornstein.[12]

where S_{ww} is the constant spectral density of an ideal white noise random process, and $\delta(\cdot)$ is the Dirac delta function. (The paper formally defining spectral density appeared in 1930, but Uhlenbeck and Ornstein were, apparently, not aware of it. Therefore, they did not use it specifically in their definition of the coefficient on the right hand side.) They proceeded to show that the response random process, $X(t)$, $-\infty < t < \infty$, has zero mean and mean square:

$$E\left[X^2(t)\right] = \sigma_x^2(t) = \frac{\pi S_{ww}}{2\zeta \omega_n^3 m^2}$$
$$\left\{1 - e^{-2\zeta\omega_n t}\left[1 + \frac{2\zeta^2}{1-\zeta^2}\sin^2\left(\omega_d t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(2\omega_d t\right)\right]\right\} \quad t \geq 0 \quad (6)$$

and they specified that the response has a Gaussian probability distribution. In this expression, $\omega_n = \sqrt{k/m}$ is the natural frequency of the SDOF structure, $\zeta = 0.5c/\sqrt{mk}$ is the system damping factor ($\zeta \in [0,1]$), and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is its damped natural frequency. Their result demonstrates the practical result that the standard deviation of displacement response approaches a finite limit, $\sigma_X(\infty) = (\pi S_{ww})/(2\zeta\omega_n^3 m^2)$, as $t \to \infty$. The limit is reached within one percent when $(0.18/\zeta)$ cycles of response have been executed. Though Uhlenbeck and Ornstein did not express the response autocorrelation function, and they did not refer to input or response spectral density, their presentation must be considered the first to use "modern" techniques of random vibration analysis.

An example that shows the result developed by Uhlenbeck and Ornstein is provided in Figure 6. Six separate marginal PDFs of the displacement response of an SDOF structure excited by ideal white noise are shown, plotted above the normalized time versus normalized displacement plane. In the graph, time, $\tau = \omega_n t$, is normalized by the natural frequency of the SDOF structure. Displacement, $\xi = x/\sigma_X(\infty)$, is normalized by the root-mean-square response at $t \to \infty$, and the PDF is normalized as $\sigma_X(\infty) f_{X(\tau/\omega_n)}\left(\sigma_X(\infty)\xi\right)$, where $f_{x(t)}(\cdot)$ is the PDF of displacement response.

Many more contributions have been made to the theory of random vibration, and some of them will be summarized later, in the section titled "Random Vibration in Engineering Structural Dynamics."

Weiner and the Definition of Spectral Density

It is quite unlikely that the mathematical theory of random vibration would have realized the widespread use it enjoys, today, if Norbert Weiner had not developed the idea of spectral density.[13] (He named it spectral intensity. Today, it is also called mean square spectral density, power spectral density, and by other descriptive names.) His paper is a 242-page, mathematically dense, presentation that formally defines spectral density and pursues many other topics including a definition of cross-spectral density, and other spectral quantities. He attributed his inspiration to papers by Schuster.[14-16] Schuster had made several efforts to define a form of spectral density, but had never defined it using a limit as did Weiner, and as we do today.

Strictly speaking, the spectral density exists only for stationary

random processes; i.e., random processes in a statistical steady-state whose realizations extend from the infinite past to the infinite future. No such random source actually exists; however, many random phenomena realize steady-state signals over relatively long periods of time. Those are the sources we seek to model. Weiner's definition of spectral density is based on the idea of autocorrelation function. The autocorrelation function of a random process is the average value, over all time, of the product of the process at time t, times the process at time $t + \tau$. For a stationary random process, this average is a function of τ, only. The mathematical definition is:

$$R_{XX}(\tau) = E[X(t)X(t+\tau)] = \lim_{t\to\infty}\frac{1}{2T}\int_{-T}^{T}x(t)x(t+\tau)dt \quad -\infty < \tau < \infty \quad (7)$$

where $x,(t), -\infty < t < \infty$ is a random process realization, i.e., a signal that comes from a stationary random source. The definition makes the assumption that a realization of a stationary random process representative of all other realizations is used; the assumption amounts to an assumption that the random source is ergodic.

The autocorrelation function, while indispensible in many frameworks, is a blunt tool for the interpretation of the character of a random process. For example, Figures 7a and 7b show a segment of a finite duration realization of a structural acceleration response random process and its estimated autocorrelation function. The time history of Figure 7a does not clearly reveal the character of the random source, and neither does the estimated autocorrelation function of Figure 7b. Weiner defined the spectral density as the Fourier transform of the autocorrelation function.

$$S_{XX}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}R_{XX}(\tau)e^{-i\omega\tau}d\tau \quad -\infty < \omega < \infty \quad (8a)$$

where ω is circular frequency in units of rad/sec, and $i = \sqrt{-1}$ is the imaginary unit. (This formula defines the two-sided spectral density, used in random vibration analysis. The one-sided spectral density is used to present experimental results, and most frequently, to plot spectral densities; it is written as a function of cyclic frequency, with units of cycles/sec = Hertz.) Weiner then proceeded to develop many of its features. Among other things, the spectral density is non-negative. The autocorrelation function is recoverable from the spectral density because the operation of Fourier transformation is invertible.

$$R_{XX}(\tau) = \int_{-\infty}^{\infty}S_{XX}(\omega)e^{i\omega\tau}d\omega \quad -\infty < \tau < \infty \quad (8b)$$

Because of this and because the autocorrelation function is defined (see Eq. 7), $R_{XX}(\tau) = E[X(t)X(t+\tau)]$, it follows that:

$$R_{XX}(0) = E\left[X^2(t)\right] = \int_{-\infty}^{\infty}S_{XX}(\omega)d\omega \quad (8c)$$

That is, the area under the spectral density curve equals the mean square of the random process. Beyond this preliminary conclusion, Eq. 8c also shows that the mean square is built up from components in different frequency ranges, and this fact can be used to draw important conclusions about a random motion. (See, for example, the following paragraph.)

Figure 7c shows the estimated spectral density of the random process that is the source of the time history shown in Figure 7a. The area under the spectral density curve is the mean square of the random process, $\sigma_X^2 = 1.23g^2$, and the value of the autocorrelation function at $\tau = 0$. The graph of the spectral density is a sequence of peaks. Because the signal upon which the spectral density estimate is based is the acceleration response of a structure, the peaks indicate mean square response signal content at the modal frequencies of the structure. The spectral density is easily interpretable; the contribution to total mean square response by each mode can be directly approximated.

An important historical note that relates to spectral density involves, so-called, ideal white noise. Ideal white noise is a random process that possesses a spectral density that is constant over all frequencies (equal energy per unit frequency), from minus infinity to infinity. (The terminology comes from a reference to the spectrum of light.) Of course, such a random process has spectral density with infinite area, and therefore, infinite mean square signal content; it cannot exist, in nature. It does excite measures of response in some structures with finite signal content, therefore, it is often used in simple modeling applications.[3]
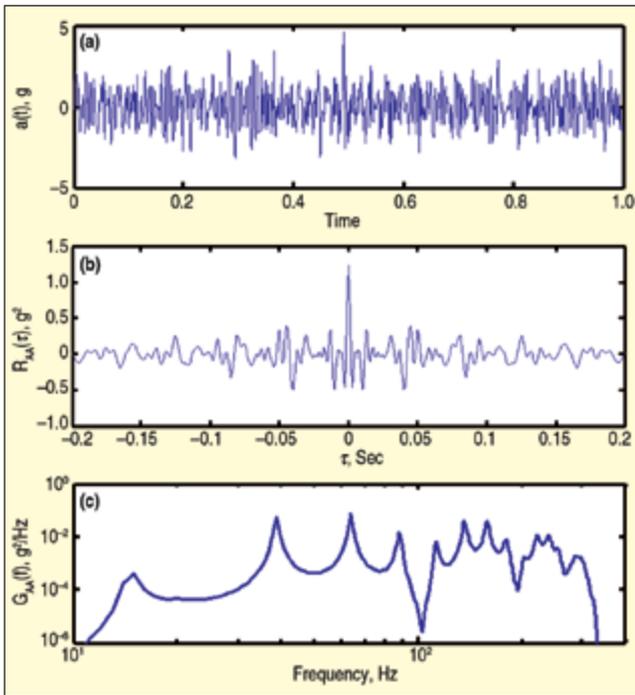
Figure 7. (a) Realization of a stationary random process; (b) estimated autocorrelation function of the source of the random signal in Figure 7a; (c) estimated spectral density function of the source of the random signal in Figure 7a.

Weiner developed many other measures of stationary random process behavior and many more features of those measures and stationary random processes, themselves. Some will be mentioned, briefly, in the following section. Is should be noted that in 1934 Khintchine[17] published a paper in which he independently defined spectral density. For that reason, Eqs. 8a and 8b are called the Weiner-Khintchine relations.

Random Vibration in Engineering Structural Dynamics

The modern field of random vibrations of mechanical systems and probabilistic structural dynamics, in general, has gained importance as the awareness that real mechanical environments are stochastic has broadened. Today random vibration analyses are performed frequently and in practical settings, usually within the framework of a commercial finite element code. Commercial finite element codes include, in general, rather limited capabilities to perform probabilistic structural dynamic analyses, at least without much pre- and post-processing. The most common analyses are those wherein the auto- (and, perhaps, cross-) spectral densities of stationary excitation random processes are specified, and response auto- and cross-spectral densities are computed.

Steven Crandall is normally credited as the person (in the United States) who made the topic of random vibrations of mechanical systems accessible to practicing engineers. He organized a summer program at the Massachusetts Institute of Technology devoted to presentations on the fundamental topics in random vibrations. The presentations are published in Reference 18 and cover analysis of and design for random vibration, testing, data analysis, spectral density estimation, and other topics.

Chapters 1, 2, and 4[18], by Crandall[19], Siebert[20], and Crandall[21], cover deterministic mechanical vibrations, the description of random processes, and a basic introduction to random vibration. In Chapter 4, Crandall combined the ideas of the previous chapters to obtain, what he called, the fundamental relation of random vibrations; although the result is much more general, he wrote the relation for a linear SDOF structure. Given that a structure is excited by a stationary, Gaussian random input, $Q(t)$, $-\infty < t < \infty$, with spectral density $S_{QQ}(\omega)$, $-\infty < \omega < \infty$, and given that the structure has frequency response function (FRF), $H(\omega)$, $-\infty < \omega < \infty$, the spectral density of the structural response is:

$$S_{XX}(\omega) = |H(\omega)|^2 S_{QQ}(\omega) \quad -\infty < \omega < \infty \quad (9)$$
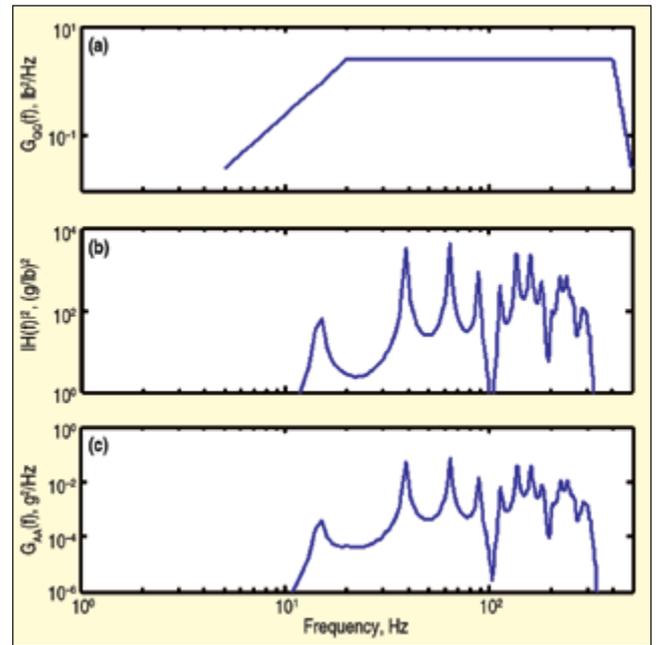

Figure 8. (a) Spectral density of a stationary, Gaussian force random process that is input to a structure; (b) modulus squared of the frequency response function of a single-input/single-output structure (FRF relates input force to response acceleration); (c) spectral density of the random acceleration structural response.

where $|\cdot|^2$ denotes the modulus squared of the complex FRF. In order to maintain an intuitive quality in the development, Crandall wrote the relation for the force-input/displacement-output FRF of an SDOF system, but, in fact, the relation is valid for any (appropriate) measure of input, any measure of response, and any single-input/single-output (SISO) FRF.

Reconsider the example presented in the previous section. The single input to the structure was a stationary, Gaussian random force with the spectral density shown in Figure 8a. The FRF of the structure has the modulus squared shown in Figure 8b. The FRF relates input force to acceleration response. The spectral density of the acceleration response is shown in Figure 8c.

The remainder of Crandall's Chapter 4 and the other chapters in Reference 18 developed many more features of random vibration excitations and responses, and many more techniques for analysis of response behaviors. All are worthy of attention, even in today's environment. One chapter that deserves special attention is the one covering "Instrumentation for Random Vibration Analysis," by Rona.[22] In it he describes an intuitive framework for understanding the meaning of spectral density, and a means for estimating spectral density from analog signals using that framework. The method he described was used in the laboratory until the late 1960s to estimate spectral density, then modified into a digital framework for spectral density estimation after the fast Fourier transform[23] (FFT) became available.

The fundamental relation of random vibration, Eq. 9, was popularized among practicing engineers by Crandall, but it had been developed in many forms by others who preceded him. The earliest reference, found by this author, to a relation like the fundamental relation of random vibrations appears in a paper written by Carson.[24] His version of spectral density was defined in the frequency domain, but without taking the limit as $t \to \infty$. In a paper written in 1931,[25] Carson defined the spectral density as we do today, taking the limit. His work dealt with the effects of noise on an electrical communications system, and in his first paper, he named the spectral density the energy spectrum of random interference. In his latter paper, Carson named the spectral density the energy-frequency spectrum.

Wiener also pursued the fundamental relation of random vibration in his 1930 paper.[13] He did so in a far-ranging section entitled "Spectra and Integration in Function Space." Without writing the formula, he stated the relation that is most fundamental to the modern practice of random vibrations of linear mechanical systems. He

stated the result for a white noise excitation in three ways; here are two of them. First, he wrote, "the spectral density of (random linear system response) is half the square of the modulus of the Fourier transform of (the system impulse response function)." Later, "if a linear resonator is stimulated by a uniformly haphazard sequence of impulses, each frequency responds with an amplitude proportional to that which it would have if stimulated by an impulse of that frequency and of unit energy."

The excitation he is referring to is a type of white noise. Wiener thus provided what is probably the original expression of the fundamental relation including the use of limits in the definition of spectral density. Wang and Uhlenbeck[26] appear to be the first authors to write the fundamental relation of random vibration in matrix form.

Although the volume edited by Crandall served as a popular and intuitive introduction to random vibration, it was preceded by some research and development activity in the engineering community. For example, Miles[28] wrote a paper in 1954 on the subject of fatigue of randomly excited structures (perhaps the first on that subject) in which he suggested a widely cited approximation. He suggested that the spectral density of one mode of the displacement or velocity response of a randomly excited, lightly damped structure can be approximated as the spectral density of the response to white noise. In equation form:

$$S_{XX}(\omega) = |H(\omega)|^2 S_{QQ}(\omega_n) \quad -\infty < \omega < \infty \quad (10)$$

where $S_{XX}(\omega), -\infty < \omega < \infty$ is the spectral density of a response random process, $S_{QQ}(\omega_n)$ is the spectral density of an excitation random process evaluated at the fundamental frequency ω_n of an SDOF structure, and $|H(\omega)|^2, -\infty < \omega < \infty$ is the modulus squared of the FRF of an SDOF structure. Thomson and Barton[29] extended Miles's approximation to multiple-degree-of-freedom (MDOF) structures in 1957.

Y. C. Fung wrote a paper in 1955[30] that was far ahead of its time. In it he developed a means for treating nonstationary random vibration. The objective of his main development was to obtain the formula for the expected value of the nth power of the nonstationary random response of a linear structure given information on the character of a nonstationary random excitation. He needed to establish a means for characterizing a nonstationary random process and did so in both the time and frequency domains.

There were other developments in random vibration prior to 1958 and many of them are summarized in Reference 31.

Current Issues

There are many topics in the field of random vibration where much potential for development remains. Among those are the practical characterization of nonstationary random processes, including excitations and responses, the characterization of structures, themselves, as random, and the random vibration of nonlinear structures.

Nonstationary random Processes. Interest in nonstationary random processes arose as early as the 1940s when they were required for use in quantum mechanics investigations.[32] Several approaches for modeling nonstationary random processes were proposed in the following years, including the one, already mentioned, by Fung.[30] However, it was 1965 before Priestley[33,34] defined a framework that is today considered the fundamental definition of nonstationary random processes. Formally, the nonstationary random process, $X(t), t \in T$, is defined

$$X(t) = \int_{-\infty}^{\infty} A_X(\omega, t) e^{i\omega t} dZ(\omega) \quad t \in T \quad (11)$$

where the form is known as a stochastic integral. (Definition of a similar, stationary, form goes back to the famous paper by Wiener.[13]) The function, $A_X(\omega, t), -\infty < \omega < \infty, t \in T$, depends on frequency ω and time t. The function $e^{i\omega t}$ is Euler's harmonic function. The increment of the integral $dZ(\omega)$ is a complex, random, infinitesimal. Its real and imaginary parts are usually taken to be zero-mean, uncorrelated, Gaussian random variables defined so that $E[|dZ(\omega)|^2] = S_{zz}(\omega)d\omega$, where $S_{zz}(\omega), -\infty < \omega < \infty$ is a spectral density.
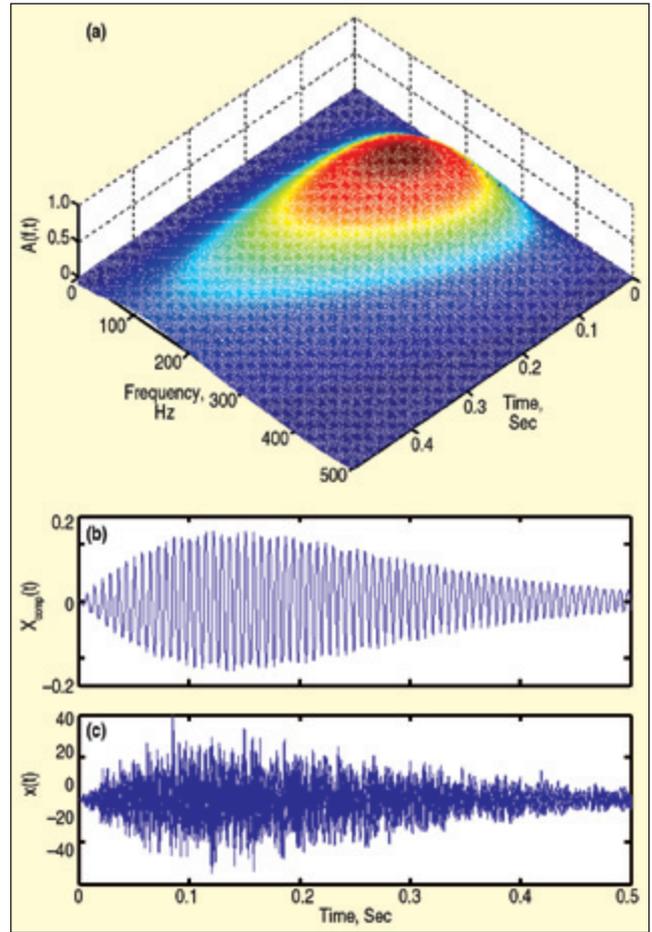

Figure 9. (a) Surface plot to the amplitude/frequency term A_{kj} in a stochastic representation of a nonstationary random process; (b) one component of a nonstationary random process realization with A_{kj} from Figure 9a; (c) one realization from a nonstationary random process with A_{kj} from Figure 9a.

In addition, the differentials $dZ(\omega_1)$ and $dZ(\omega_2)$ are uncorrelated when $\omega_1 \neq \omega_2$. The quantity $S_{XX}(\omega, t) = |A_X(\omega, t)|^2 S_{zz}(\omega), -\infty < \omega < \infty, t \in T$, is the evolutionary spectral density of the nonstationary random process. At a given time t it describes the distribution of mean square signal content of the random process.

The expression in Eq. 11 is abstract, and it might be unfamiliar to all but specialists in the field of random processes, but writing it in computational form clarifies it greatly. Let $X_j, j = 0, \ldots, n - 1$ denote a zero-mean, nonstationary random process with discrete parameter; i.e., defined at times $t_j = j\Delta t, j = 0, \ldots, n - 1$. A realization of this random process might be generated using the formula:

$$x_j = \sum_{k=-n/2+1}^{n/2} A_{kj} e^{i2\pi jk/n} \Delta z_k \quad j = 0, \cdots, n-1 \quad (12)$$

where A_{kj} is the discrete time (j) and discrete frequency (k) equivalent of $A_X(\omega, t)$, $e^{i2\pi jk/n}$ corresponds to $e^{i\omega t}$, and the $\Delta z_k, k = -n/2 + 1, \ldots, n/2$ are complex-valued samples from a normal source. The signal covers the frequency interval $[-1/2\Delta t, 1/2\Delta t]$. The real and imaginary parts of each Δz_k come from uncorrelated random variables, and the variates that make up Δz_{k1} and Δz_{k2} all come from uncorrelated sources when $k_1 \neq k_2$. We would choose $\Delta z_{-k} = \Delta z_k^*$, and $A_{-kj} = A_{kj}^*$ if we wish for the realization, $x_j, j = 0, \ldots, n - 1$, to be a real-valued function of time.

To picture how signals are generated with Eq. 12, define $n = 512$, $\Delta t = 0.001$ sec, and $t_j = j\Delta t, j = 0, \ldots, n - 1$; because of the time increment and the number of steps in the time signal, the frequency increment is $\Delta f = 1/n\Delta t = 1.953$ Hz, and $f_k = k\Delta f, k = 0, \ldots, n/2 + 1$. Define:

$$A_{kj} = 60 a_j t_j f_k \left[1 - \cos(2\pi f_k / 500)\right] e^{-5t_j} e^{-\alpha_j f_k} \quad k = 0, \cdots, 256, j = 0, \cdots, 511 \quad (13a)$$

where

$$\alpha_j = 0.004 = 0.032 t_j \quad j = 0, \cdots, 511 \quad (13b)$$

and

$$a_j = \alpha_j e^{-(\alpha_j t_j + 1)} \quad j = 0, \cdots, 511 \quad (13c)$$

The surface plot of the function A_{kj}, $k = 0,...,256$, $j = 0,...,511$, is shown in Figure 9a. (Components with positive frequency index, only, are shown.) Consider the component at $k = \pm99$. At that frequency index (and for this realization) $\Delta z = 0.1872 + 0.0841i$ is the complex variate drawn from a normal source. The single frequency component at $k = \pm99$ is shown in Figure 9b. It is simply a slowly modulated harmonic function with amplitude that comes from a random source. When all 257 components are summed, the result is the signal shown in Figure 9c. The random vibration response of a linear structure to a nonstationary input with this form is easily established. Prior to Priestley's development of the evolutionary spectrum model, another means for modeling nonstationary random processes was under development[35,36]; it is the Karhunen-Loeve expansion (KLE).[37-39] It is a more efficient framework for representing nonstationary random processes. It expresses a discrete parameter random process $X_j, j = 0,...,n-1$, as:

$$X = VW^{1/2} U_0 + \mu_X \cong vw^{1/2} U + \mu_X \qquad (14)$$

where X is the $n \times 1$ column vector of random variables in the random process, V is the $n \times n$ matrix of orthonormal eigenvectors of the covariance matrix of X, W is the $n \times n$, diagonal matrix of non-negative eigenvalues of the covariance matrix of X, U_0 is an $n \times 1$ column vector of zero-mean, unit variance, uncorrelated random variables that underlie the representation, and μ_X is the $n \times 1$ vector containing the mean of the random process. The efficiency of the KLE is that it can be approximated using a fraction of its terms. Assume that the diagonal terms in W are arranged in descending order from top to bottom. Only those terms, say, $m < n$, that contribute "substantially" to the sum of the diagonal are retained in the $m \times m$ matrix, w. The matrix $w^{1/2}$ contains the amplitudes of the representation. The corresponding vectors of V are retained in the $n \times m$ matrix v. The matrix v contains the characteristic shapes of the representation. U is an $m \times 1$ vector of random variables. These are the randomizing factors of the representation.

Normally, the mean vector and covariance matrix of a nonstationary random process are not known from theoretical considerations, but data from a random source can be measured. These data can then be used to estimate the mean vector and covariance matrix of the random process. The eigenvalue decomposition of the covariance matrix can be computed to estimate the KLE parameters v and w. If the joint distribution of the random variables in U can be approximated, then the parameters and vector of random variables can be used in Eq. 14 to generate realizations of the random process $X_j, j = 0,...,n-1$.

For example, 20 measured realizations of a short-duration random process (shock) are available; they are shown in Figure 10a. The surface plot of the covariance matrix estimated from the measured samples is shown in Figure 10b. (The "hashy" character of the surface is a reflection of estimation noise.) The estimated covariance matrix is decomposed using an eigenvalue analysis and eigenvalue analysis, m = 10 components are retained in the KLE approximation. The retained shape functions are shown in Figure 10c. The square roots of the (retained) eigenvalues are shown in Figure 10d. The random vector U is assumed to have a jointly Gaussian distribution. Forty realizations of the random process were generated using the estimated parameters with 40 generated 10×1 vectors, U, of uncorrelated, standard, normal, random variables and the form of Eq. 14. They are shown in red in Figure 10e along with the original, measured random process realizations. They show that the KLE can generate an accurate representation of a nonstationary random process source.

The KLE provides a good representation of many nonstationary random processes whether they are functions of time or another parameter. The important thing to remember about the KLE is that an approximation to it can be estimated whenever a collection of realizations from a random source is available. Therefore, it can be very useful in Monte Carlo applications.

Stochastic Structure Modeling. Most of the earliest practitioners of random vibration analysis recognized that physical structures are random, as well as the inputs that excite them. However, they considered the degree of randomness of excitations to normally be greater than that of the excited structures, and they concentrated on developing a theory of random vibration that considered structural properties deterministic. Still, some early researchers considered
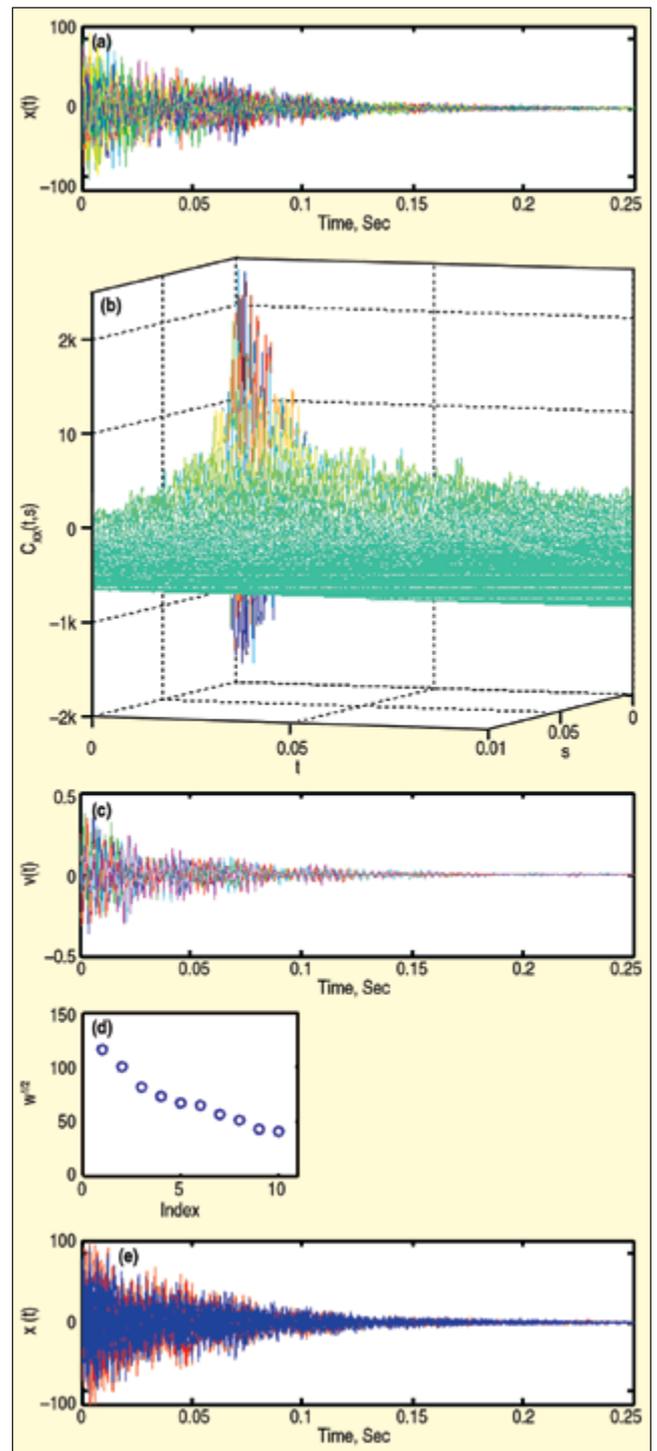

Figure 10. (a) Twenty realizations from a nonstationary random source; (b) estimated covariance of the random source of the signals in Figure 10a; (c) estimated shape functions of the KLE of the source of signals in Figure 10a; (d) estimated amplitudes of the KLE of the source of the signals in Figure 10a; (e) generated realizations (red) from the approximate source of the realizations in Figure 10a (blue), assuming that U is jointly Gaussian.

structural randomness and its effects.[40-42]

Several researchers sought a general theory for analysis of stochastic structures. Ghanem and Spanos[37] developed a technique with a good potential for wide use because their technique is based on the finite element method. The technique, called stochastic finite elements (SFE), is too complicated to explain in fine detail, here, but the following steps indicate how it works (a simple example is presented later):

1. Write the partial differential equation (PDE) governing the structure of interest. The equation includes (a) independent variables like time and spatial coordinates, (b) dependent variables – the variables to be solved for – like displacement and acceleration,

Figure 11. Shallow beam with random tip load.

and (c) parameters that are usually written as coefficients on the left side of the PDE, like area, modulus of elasticity, mass, moment of inertia, etc.

2. Represent the PDE coefficients that are random as a random field in the KLE framework. (This can be done directly, with the use of finite elements, for one-, two-, and three-dimensional fields.) The randomizing coefficient of the KLE – U in Eq. 14 – may be Gaussian or non-Gaussian. The randomness is modeled as Gaussian, here, and is denoted ξ, where ξ is a vector of standard normal random variables. Denote the random coefficient $\alpha(u, \xi)$, where u is a spatial coefficient, and let n_a denote the number of terms retained in the KLE approximation.

3. Represent the dependent variable in the KLE framework. The randomizing coefficient of the KLE will normally not be Gaussian, and Ghanem and Spanos recommend that the random vector U be represented with a polynomial chaos. The polynomial chaos is a Hermite polynomial expansion written in standard normal random variables. Of course, the random variables that appear in the coefficient expression, above, also appear in the solution expression, here, because the system response depends on the random features of the system characteristics. Denote the response $x(\alpha(u,\xi), t)$, and let n_x denote the number of terms retained in the polynomial chaos approximation.

4. Perform a finite element expansion on the KLE shape functions of the coefficient, $\alpha(u,\xi)$ and the shape functions of the response polynomial chaos, $x(\alpha(u,\xi), t)$, to obtain a discrete set of stochastic governing equations. The governing equation has dimension $N \times N$, where N is the number of degrees-of-freedom in the finite-element model.

5. Use the characteristics of the standard normal random variables and the expectation operation on the governing equation to obtain a collection of simultaneous, ordinary differential equations governing the shape functions and amplitudes of the solution, $x(\alpha(u,\xi), t)$. The number of equations relates to the desired accuracy of the solution approximation, embodied in n_a and n_x.

6. Solve the collection of matrix ODEs to obtain a specific expression for $x(\alpha(u,\xi), t)$. Use the expression to infer the characteristics of system response.

An example elucidates how the SFE method can be used in many different, sometimes very simple, ways. Consider a slender beam, shown in Figure 11, with a random vibration load on the tip. The beam is excited in the vertical plane and responds in the vertical plane. We seek, in the following, to excite realizations of a random beam with realizations of a nonstationary random process; this is a Monte Carlo analysis.

The Euler-Bernoulli version (applicable to shallow beams without shear and rotation effects) of the equation governing motion of a slender beam is:

$$m\ddot{x} + \frac{\partial^2}{\partial u^2}\left[E(u)I(u)\frac{\partial^2 x}{\partial u^2}\right] = w(u,t) \qquad u \in [0,L] t \geq 0 \qquad (15)$$

Its solution is made specific through specification of boundary conditions (BC) and initial conditions (IC). We take the BC to be $x(u,t)\,|_{u=0} = 0$ and $\partial x(u,t)/\partial u\,|_{u=0} = 0$. We take the IC to be $x(u,t)\,|_{t=0} = 0$ and $\partial x(u,t)/\partial t\,|_{t=0} = 0$.

The load is applied at the tip, only, therefore, $w(u,t) = \delta(u-L) Q(t), t \geq 0$. $Q(t)$ is simulated at discrete times with, $Q_j, j \in [0,\ldots,511]$, the type of nonstationary random process modeled in the previous example; the load is Gaussian. The load is defined at times $t_j = j\Delta t, j = 0,\ldots,511$, with $\Delta t = 1 \times 10^{-4}$ sec. Twenty realizations of the load random process are shown by the curves in Figure 12; as many realizations of the excitation as desired can be generated from the KLE model.

We could carry out the operations described in the six steps listed above; however, we intervene early in the process to simplify the analysis. Let $\alpha(u) = E(u)I(u), u \in [0, L]$, denote a Gaussian random field with mean $\mu_\alpha(u), u \in [0, L]$, and autocovariance function

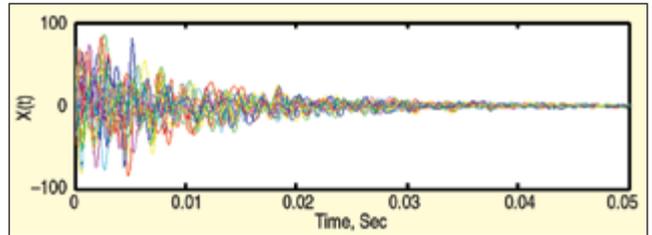

Figure 12. Twenty realizations of short-duration random load applied to the tip of beam.
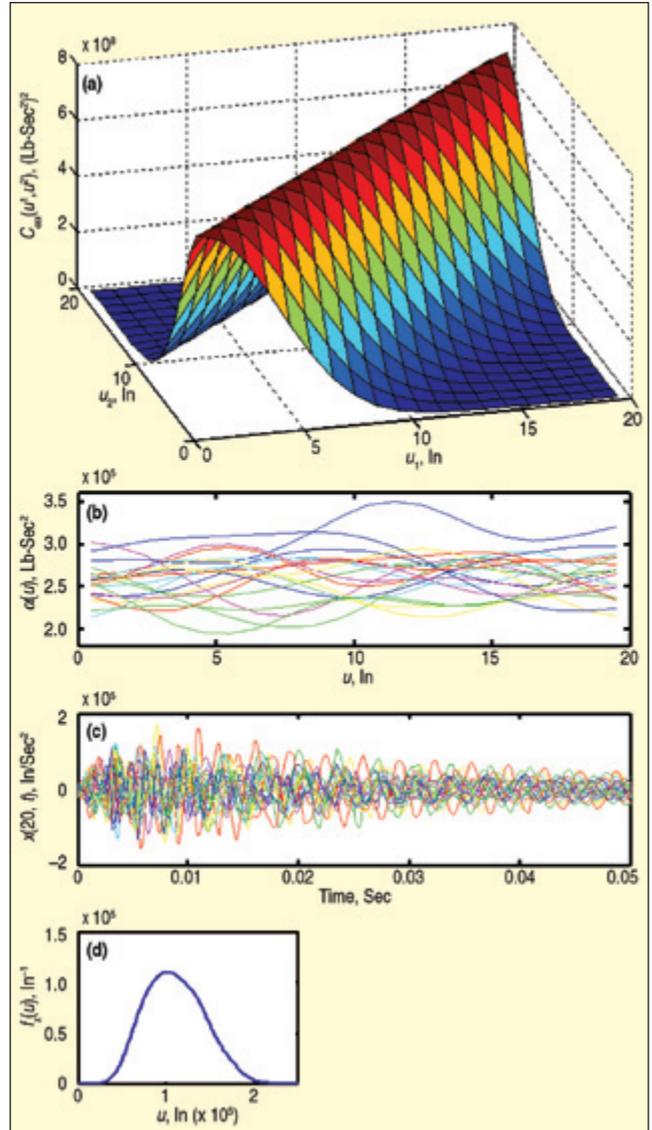

Figure 13a. Autocovariance function, $C_{\alpha\alpha}(u_1,u_2)$ $u_1,u_2 \varepsilon$ [0, L], of the random field $\alpha(u)$, $\upsilon u[0, L]$; (b) realizations drawn from the KLE approximation of the random field with autocovariance in Figure 13a, and mean $\mu_\alpha(u) = 2.667 \times 10^5$ lb-in^2; (c) 20 realizations of stochastic beam tip acceleration response (beam has $a(\upsilon) = E(u)I(u)$, $u\varepsilon[0, L]$ that is a random field; some realizations of $\alpha(u)$ are shown in Figure 13b) to a short duration random load applied to the tip of the beam. (20 realizations of the load are shown in Figure 12); (d) approximate probability density function of the peak absolute acceleration response at the tip of the beam; beam and load are stochastic, and PDF is very clearly non-Gaussian.

$C_{aa}(u_1,u_2) = \sigma^2 \exp[-\beta(u_2 - u_1)^2], u_1,u_2 \in [0, L]$. The mean is defined as a constant, $\mu_\alpha(u) = 2.557 \times 10^5$ lb-in^2. The parameters of the autocovariance are $\sigma^2 = (2.557 \times 10^4)^2$ (lb-in^2)^2. The beam length is $L = 20$ in. The finite-element discretization is $u_j = j, j \in [0,1\ldots20]$, i.e., the u_j identify the nodes of the finite-element mesh. The random field $\alpha(u)$ is defined as constant within each beam element, so the discretization for $\alpha(u)$ identifies the coefficient at locations (1/2) $(u_j + u_{j+1})$, $u_j = j, j \in [0,1\ldots19]$. The autocovariance function for the field $\alpha(u)$ is shown in Figure 13a. Some realizations drawn from the KLE model of the random field are shown in Figure 13b; as
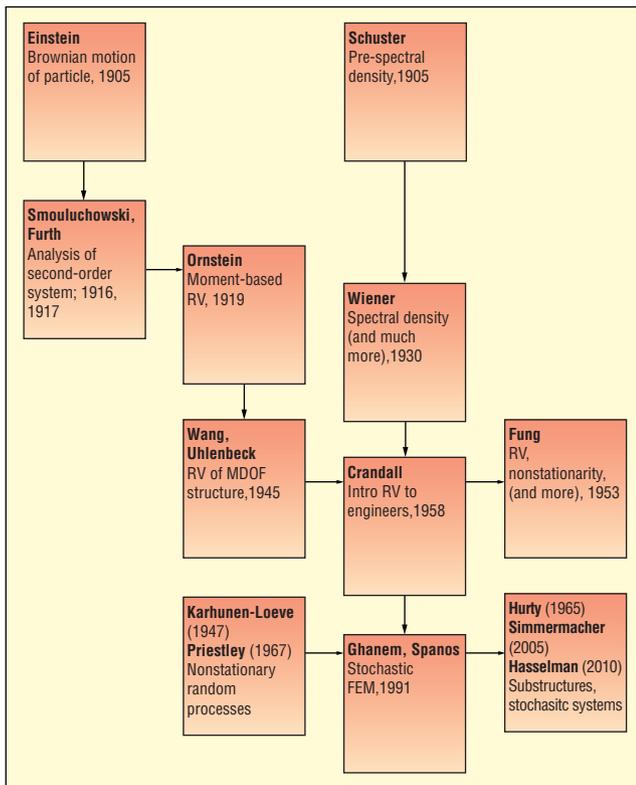
Figure 14. History of random vibration.

many realizations as desired can be drawn from the KLE of $\alpha(u)$.

For purposes of this example, we choose to generate 100 realizations of the random excitation. We generate 100 realizations of the stochastic beam, each one excited by one of the inputs. Twenty of the 100 computed acceleration responses are shown in Figure 13c. The analyst may be interested in knowing the probability distribution of some measure of the stochastic response, say, peak absolute acceleration at the tip of the beam. Figure 13d shows the kernel density estimator (KDE),[43] an approximation to the PDF) of the peak absolute acceleration based on the 100 computed responses.

The SFE method can be used in many more ways. For example, modal analysis can be directly performed on models leading to a probabilistic description of a stochastic beam. The probabilistic description of the modes can then be used to investigate random structural response.

Random Vibration of Nonlinear Structures. Throughout the history of random vibration it has been recognized that real structures are nonlinear. Some structures are only slightly nonlinear and can be treated as linear; some are so nonlinear that the nonlinearity must be accommodated in analysis. Some structures are nonlinear because of very localized effects. Hundreds of papers and texts dealing with random vibration of nonlinear structures have been written, for example.[7,44-46] Through a decade, or so, ago, treatments of nonlinear random vibrations were very specialized and not generally applicable to practical nonlinear structures.

Then it was recognized that nonlinear behavior in many, stiff structures emanates not from the behavior of the overall structure, but from the behavior of some much localized effects, for example, in mechanical joints. That is, the main components of the structure behave approximately linearly, but a few components behave nonlinearly. That realization led to procedures for the analysis of practical-scale, nonlinear structures.[47,48] The more recent developments are based on work done several decades ago.

In 1965 Hurty[49] wrote a paper describing a method for writing the equations of a linear, dynamic structure in terms of the modes of the component parts of the structure. Because structural components can often be accurately characterized in terms of a relatively small number of lower frequency modes, the method provides the potential for great efficiency in the sense that the number of equations required to characterize the structural dynamics is greatly reduced, relative the number of degrees-of-freedom (DOF) represented in the

component models. Craig and Bampton[50] rediscovered and slightly modified the method in 1968, and the method is amply described in Ref. 51. The method has come to be known as the Craig-Bampton method. The essence of the original technique is this:

1. Perform a classical, structural dynamic modal reduction on linear structural components restraining components against motion at DOF where connections occur.
2. Identify, via static analysis, deformations of component internal DOF in terms of component boundary DOF. Take the resulting deformations to represent the so-called "restraint modes." (These are included in the following transformation to help convergence of the representation.)
3. Construct a transformation matrix consisting of dynamic mode shapes from 1 and the static model shapes from 2.
4. Write the equation governing linear structural dynamics of the overall structure including linear constraint equations that tie the components together.
5. Organize the component transformation matrices into an overall structure transformation matrix to reduce the equations of motion. The number of reduced equations governing the overall structure equals the sum of the number of dynamic modes retained in the components plus the number of boundary DOF in the components. This quantity may be substantially lower than the total DOF in the structural model including all the components.

To this point, none of the development is aimed specifically at random vibration or nonlinear analysis, but the method can be used to perform random vibration computations. To extend the technique to the analysis of locally nonlinear structures Simmermacher[47] modified steps 4 and 5, above. In writing the governing structural dynamic equations for the system, he wrote constraint equations that incorporate a nonlinear restoring force. Further, he made the parameters of the nonlinear mechanical connection jointly distributed random variables. The joint distribution of the random variables is based on data from laboratory experiments. In a sense, he wrote a simple stochastic differential equation that could be solved using Monte Carlo simulation.

Hasselman[48] went further. He modeled the linear structural components as random entities, and he did so in a framework compatible with experimental modal analysis. His representation of modal randomness in structural components is second order; that is, he encapsulates a representation of randomness in the covariance matrix of modal mass, stiffness and damping. He obtains the covariance matrix of modal randomness for each component from a comparison of model-derived component modes to experimental component modes. With this representation and a mode synthesis procedure, structural randomness, as represented by the covariance matrix of random structural characteristics, is modeled.

Conclusion

The history of the mathematical theory of random vibrations of mechanical systems spans the previous century, starting with the work of Einstein, and continues to the present. The theory is essential to the representation and interpretation of realistic inputs and responses of structures. This article briefly presents some developments in that history through the present time. Although the groundwork for random vibration analysis was laid over that entire period, the work dealing with application to structural and mechanical systems started in earnest in the 1950s. Numerous papers and texts contributing to the development of random vibrations have been summarized.

For the convenience of the reader Figure 14 summarizes some of the names from the history of random vibration, the years of their contributions, and a phrase describing their contributions.

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