Statistical Properties of the Random PSD

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Control and conduct of a random vibration test is all about probabilities. What is the chance of any single PSD spectral line being less than X? More than Y? What percentage of the test time will the amplitude at any frequency be between X and Y? Well established statistical calculations answer these important questions. In this article we present various procedures for determining the statistical properties of random waveforms. Probability and confidence interval tables are introduced for common degrees of freedom.

The power spectral density (PSD) of a Gaussian random waveform can be computed using the fast-Fourier transform (FFT). The FFT is a linear transform. If it is given a Gaussian input, the output at each frequency line is a complex number with a Gaussian real part and a Gaussian imaginary part. These are squared and added together to get the PSD magnitude, a Chi-square distributed random variable with 2 degrees-of-freedom (DOF). To compute an averaged random PSD, *F* frames of time data are measured, an FFT of each frame is taken and the squared magnitudes are averaged together. As a result, the averaged PSD of a Gaussian waveform is a Chi-square distributed random variable with (*2F*) DOF. This is where the DOF averaging specification for a random vibration test comes from.

In a practical sense, this means we can directly compute confidence intervals and probabilities from the equations defining the Chi-squared distribution. The Chi-squared distribution with *n* degrees-of-freedom has a probability density function and cumulative distribution function as follows.

$$\text{PDF:} \quad f(w;n) = \frac{w^{(n-2)/2} e^{-(w/2)}}{2^{(n/2)} \Gamma(n/2)} \quad (\text{for } w = 0)$$

$$\text{CDF:} \quad F(w;n) = P(w/2, n/2)$$

where $\Gamma(a)$ is the gamma function, and $P(x,a)$ is the regularized lower incomplete gamma function.

Tolerance bands

Using this relationship and given a DOF value, we can tabulate the probability of exceeding a given dB level from the mean. If we note that the mean of a Chi-squared random variable is equal to the DOF, *n,* then using MATLAB notation:

Pr[x<dB] = gammainc(10^(dB/10)*DOF/2,DOF/2,'lower') for dB<0

Pr[x>dB] = gammainc(10^(dB/10)*DOF/2,DOF/2,'upper') for dB>0

Compounded Probability

The probability calculations give the probability for a single line being outside the tolerance band. A typical random test is composed of a broad band of frequencies, with 200, 400, 800 or more lines distributed over that band, and we are interested in knowing the probability of any *one* of those lines going outside of the tolerance or abort bands. To account for this we assume the lines are independent, and compute the probability for having all lines within the specified tolerance band. This is done by computing the probability of a single line being within the tolerance range, and then combining this probability together with all other lines. Mathematically this is done by multiplying the probabilities together. So, for a given number of lines, we can tabulate the probability of meeting the tolerance using the formula p^l where p is the probability of a single line being in-tolerance and l is the number of lines.

This calculation tells us what DOF parameter is required to achieve and maintain a stated tolerance with reasonable probability. Since the waveform is random, we can never achieve a full 100%, but some cases get very close.

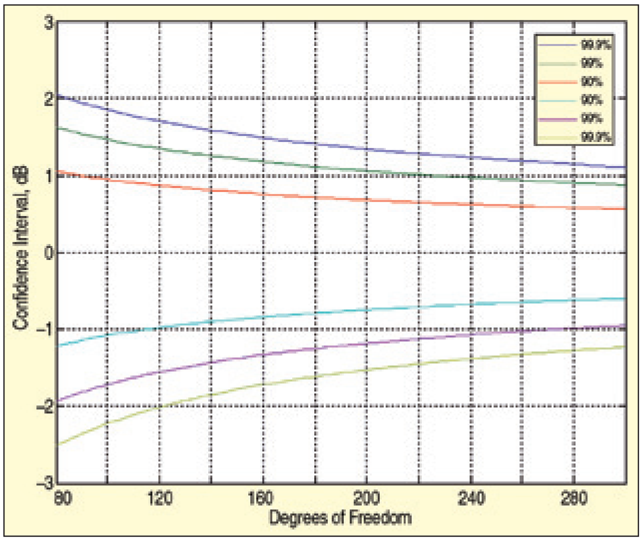

Figure 1. Confidence interval vs. degrees of freedom.

Confidence intervals

This calculation can be reversed, and we can also compute the dB levels given a probability value. This is typically expressed as a 90%, 99% or 99.9% confidence interval. The 90% confidence interval gives the range of dB values that cover the central 90% of the probability curve. In other words, there is a 5% probability of being less than the lower bound, and a 5% probability of being

Table 1. Probability, in percent, of PSD value exceeding tolerance dB for a given DOF.

DOF =	80	100	120	140	160	180	200	220	240	260	280	300
+3.0 dB	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
+2.0 dB	0.068	0.018	0.005	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
+1.5 dB	0.892	0.418	0.199	0.095	0.046	0.022	0.011	0.005	0.003	0.001	0.001	0.000
+1.0 dB	5.867	4.099	2.892	2.056	1.470	1.055	0.761	0.550	0.399	0.290	0.211	0.154
+0.5 dB	21.342	19.032	17.058	15.350	13.856	12.541	11.375	10.337	9.409	8.576	7.827	7.152
–0.5 dB	25.424	22.628	20.283	18.276	16.534	15.006	13.655	12.452	11.377	10.411	9.541	8.755
–1.0 dB	8.870	6.426	4.708	3.478	2.586	1.932	1.450	1.091	0.824	0.623	0.473	0.359
–1.5 dB	2.214	1.188	0.647	0.356	0.197	0.110	0.062	0.035	0.020	0.011	0.006	0.004
–2.0 dB	0.402	0.146	0.054	0.020	0.008	0.003	0.001	0.000	0.000	0.000	0.000	0.000
–3.0 dB	0.006	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 2. Probability, in percent, of all lines within tolerance band, 800 lines.

DOF =	80	100	120	140	160	180	200	220	240	260	280	300
±3.0 dB	95.60	99.41	99.92	99.99	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
±2.0 dB	2.32	26.92	62.49	84.21	93.83	97.64	99.10	99.66	99.87	99.95	99.98	99.99
±1.5 dB	0.00	0.00	0.11	2.69	14.28	34.71	56.01	72.64	83.76	90.60	94.62	96.94

Table 3. Probability, in percent, of all lines within tolerance band, 400 lines.

DOF =	80	100	120	140	160	180	200	220	240	260	280	300
±3.0 dB	97.78	99.70	99.96	99.99	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
±2.0 dB	15.23	51.89	79.05	91.77	96.86	98.81	99.55	99.83	99.93	99.97	99.99	100.00
±1.5 dB	0.00	0.15	3.35	16.41	37.79	58.92	74.84	85.23	91.52	95.18	97.27	98.46

Table 4. Confidence interval, in dB, of a PSD value for a given DOF.

DOF =	80	100	120	140	160	180	200	220	240	260	280	300
99.9%	2.05	1.85	1.70	1.59	1.49	1.41	1.34	1.28	1.23	1.19	1.15	1.11
99%	1.63	1.47	1.35	1.25	1.18	1.11	1.06	1.01	0.97	0.94	0.90	0.87
90%	1.05	0.95	0.87	0.81	0.76	0.72	0.68	0.65	0.62	0.60	0.58	0.56
90%	–1.22	–1.08	–0.98	–0.91	–0.84	–0.79	–0.75	–0.71	–0.68	–0.65	–0.63	–0.61
99%	–1.94	–1.72	–1.56	–1.43	–1.33	–1.25	–1.18	–1.13	–1.08	–1.03	–0.99	–0.96
99.9%	–2.52	–2.23	–2.01	–1.85	–1.72	–1.62	–1.53	–1.45	–1.39	–1.33	–1.28	–1.23

greater than the upper bound, and 90% probability of being within the stated bounds. These confidence intervals can be tabulated and plotted as a function of DOF.

Windowing and Overlap

These calculations, with DOF equal to two times the number of frames, assume independent frames of data. When overlapping frames and window functions are used, the frames are no longer independent, and the effective DOF of the average will not be simply a multiple of the number of frames. The effective DOF can be estimated numerically by simulating data with the specified overlap and window function applied, computing the statistics of the resulting PDF, and comparing those statistics to a Chi-squared distribution with a known DOF.

As a rule of thumb, for 50% overlap and with a window function, you still achieve a DOF of nearly two times the number of frames averaged together. For 75% overlap, you achieve a DOF of about one times the number of frames. For a given time interval, the practical result of this is that you can double the effective DOF by using 50% overlap. S|V

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