Proportional Damping from Experimental Data

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In most mechanical structures, we assume that there are several damping mechanisms at work, but they may be difficult to identify and even more difficult to model. Because of this difficulty, damping forces are usually not included in a finite-element analysis (FEA) model of a structure. Nevertheless, viscous damping can be modeled by assuming that the viscous damping matrix is a *linear combination* of the mass and stiffness matrices. This is called *proportional* damping.

To model proportional viscous damping, two constants of proportionality must be determined, one for the mass and one for the stiffness matrix. In this article, a least-squared-error relationship between experimental modal frequency and damping and the proportional damping constants of proportionality is developed. This formula is then used to calculate the constants of proportionality from experimental modal parameters. The modal parameters of an FEA model with proportional damping are then compared with the original experimental modal parameters.

Viscous Damping

All experimental resonant vibration data is characterized by a decaying sinusoidal response when all forces are removed from the structure. The overall response is modeled as a summation of responses, each one due to a mode of vibration. Each modal contribution is itself a decaying sinusoidal waveform. The decay envelope for each mode is modeled with a decreasing exponential function, and the decay constant in the exponent is called the modal damping coefficient. It is also called the half-power point, or 3-dB point damping.

It can be safely assumed, at least in earth's atmosphere, that the dominant damping mechanism for most structures is the viscous damping of the surrounding air. It is also assumed that viscous damping can be modeled using a linear viscous damping term. This term, in which the dissipative forces are proportional to surface velocities, is used to model the damping forces in the differential equations of motion for a structure.

The differential equations for a vibrating structure with viscous damping are written as;

$$[M]\{\ddot{x}(t)\}+[C]\{\dot{x}(t\}+[K]\{x(t)\{=\{f(t)\} \tag{1}$$

where:

$[M]$ = mass matrix (n by n)
$[C]$ = viscous damping matrix (n by n)
$[K]$ = stiffness matrix (n by n)
$\{\ddot{x}(t)\}$ =accelerations (n vector)
$\{\dot{x}(t\}$ = velocities (n vector)
$\{x(t)\}$ = displacements (n vector)
$\{f(t)\}$ = external forces (n vector)
n = number of degrees of freedom of model

Equation 1 is a force balance with internal (inertial, dissipative, and restoring) forces on the left-hand side and the external forces on the right-hand side. Equation 1 describes the linear, stationary, viscously damped, dynamic behavior of a structure. To construct an FEA model, the mass $[M]$ and stiffness $[K]$ matrices are synthesized from the geometry and material properties of the structure. Furthermore, in most FEA practice today, the damping term is assumed to be zero. That is, the damping forces are not modeled.

Modes of Vibration

The frequency domain version of Eq. 1 is commonly used as the basis for determining the modes of vibration of the structure. Modes are solutions to the homogeneous form of this equation, which is written as:

$$\left([M]p^2+[C]p+[K]\right)\{\phi\}=\{0\} \quad p=-\sigma+j\omega \tag{2}$$

Each non-trivial solution of this matrix equation consists of a pole location, p (also called an eigenvalue) and a mode shape, $\{\phi\}$ (also called an eigenvector). Each complex pole is made up of both the damping decay constant (σ) and the damped natural frequency (ω).

Proportional Damping Matrix

A proportional damping matrix is assumed to be a linear combination of the mass and stiffness matrices. That is, the viscous damping forces are assumed to be proportional to the inertial and restoring forces, as represented in the following equation;

$$[C]=\alpha[M]+\beta[K] \tag{3}$$

where:

α = constant of mass proportionality
β = constant of stiffness proportionality

If the two constants (α and β) can be determined, then all of the terms in Eq. 2 are known, and the modes of the damped FEA model can be calculated. The question then becomes, **How can α and β be determined to reflect the damping of a real structure?**

Proportional Damping Coeficients

Modal frequency and dampening estimates are routinely determined from experimental data using modern modal testing and analysis methods. Experimental forced vibration data are commonly obtained in the form of a set of frequency response functions (FRFs). An FRF is a special form of a transfer function. Its numerator is the Fourier spectrum of a structural output (acceleration, velocity, or displacement response), and its denominator is the Fourier spectrum of the input (the force that caused the response).

Experimentally derived frequency and damping estimates are obtained from one or more FRFs by curve fitting them using an analytical model that includes frequency and damping as unknown parameters. A set of experimental modal analysis (or EMA) frequency and damping estimates is therefore obtained for all modes in the frequency band of the FRF measurements.

The relationship between multiple EMA frequency and damping estimates and the coefficients (α and β) can be derived from Eq. 2. Substituting Eq. 3 into Eq. 2 and rearranging terms gives:

$$\begin{aligned}
&\left([M]p^2+\left(\alpha[M]+\beta[K]\right)p+[K]\right)\{\phi\}=\{0\}\\
&\left([M]\left(p^2+\alpha p\right)+[K](\beta p+1)\right)\{\phi\}=\{0\}\\
&\left([M]\left((-\sigma+j\omega)^2+\alpha(-\sigma+j\omega)\right)+[K]\left(\beta(-\sigma+j\omega)+1\right)\right)\{\phi\}=\{0\}\\
&\left([M]\left(\left(\sigma^2-\omega-j2\sigma\omega\right)+(-\alpha\sigma+j\alpha\omega)\right)+[K]\left(\beta(-\sigma+j\omega)+1\right)\right)\{\phi\}=\{0\}\\
&\left([M]\left(\left(\sigma^2-\omega-\alpha\sigma\right)+j(-2\sigma\omega+\alpha\omega)\right)+[K]\left((-\sigma\beta+1)+j\beta\omega\right)\right)\{\phi\}=\{0\}
\end{aligned} \tag{4}$$

Notice that Eq. 4 also has the same form as an equation for an undamped FEA model. A known property of the mode shapes $\{\phi\}$ of an undamped structure model is that they are real valued, also called normal modes.

Because $\{\phi\}$ is real valued, the real and imaginary parts of Eq. 4 are uncoupled, and therefore can be written as separate equations:

$$\begin{aligned}
&\left([M]\left(\sigma^2-\omega^2-\alpha\sigma\right)+[K](-\sigma\beta+1)\right)\{\phi\}=\{0\}\\
&\left([M](-2\sigma\omega-\alpha\omega)+[K]\beta\omega\right)\{\phi\}=\{0\}
\end{aligned} \tag{5}$$

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Putting these equations into the standard form for an un-damped structure;

$$\left(\frac{\left(\sigma^2-\omega^2-\alpha\sigma\right)}{\left(-\sigma\beta+1\right)}[M]+[K]\right)\{\phi\}=\{0\}$$
$$\left(\frac{\left(-2\sigma+\alpha\right)}{\beta}[M]+[K]\right)\{\phi\}=\{0\} \tag{6}$$

Both of these equations must be satisfied for a proportionally damped structure. Solutions to these equations are unique poles (or eigenvalues), and the coefficients of the mass matrix can be equated to each of the poles. The equation for each pole can be written as:

$$\frac{\left(\sigma^2-\omega^2-\alpha\sigma\right)}{\left(-\sigma\beta+1\right)}=\frac{\left(-2\sigma+\alpha\right)}{\beta}=-\Omega^2 \tag{7}$$

where;

$\Omega^2 = (\sigma^2 + \omega^2)$ = an undamped natural frequency squared.

This provides a single equation with two unknowns in it (α and β):

$$2\sigma=\alpha+\beta\Omega^2 \tag{8}$$

or

$$2\sigma=\alpha+\beta\left(\sigma^2+\omega^2\right) \tag{9}$$

Equation 9 can be used together with experimentally derived estimates of frequency and damping for two or more modes to compute the proportional damping constants, α and β.

Least-Squared-Error Solution

Given a set of EMA frequencies and damping for n modes (n>=2), n equations can be written:

$$2\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \vdots\\ \sigma_n\end{Bmatrix}=\begin{bmatrix}1 & \Omega_1^2\\ 1 & \Omega_2^2\\ \vdots & \vdots\\ 1 & \Omega_n^2\end{bmatrix}\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix} \tag{10}$$

Since there are only two unknowns, this is an overspecified set of equations. The least-squared-error solution of these equations is written as:

$$2\begin{bmatrix}1 & 1 & \cdots & 1\\ \Omega_1^2 & \Omega_2^2 & \cdots & \Omega_n^2\end{bmatrix}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \vdots\\ \sigma_n\end{Bmatrix}=\begin{bmatrix}1 & 1 & \cdots & 1\\ \Omega_1^2 & \Omega_2^2 & \cdots & \Omega_n^2\end{bmatrix}\begin{bmatrix}1 & \Omega_1^2\\ 1 & \Omega_2^2\\ \vdots & \vdots\\ 1 & \Omega_n^2\end{bmatrix}\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix} \tag{11}$$

or:

$$2\begin{Bmatrix}\sum_{i=1}^{n}\sigma_i\\ \sum_{i=1}^{n}\sigma_i\Omega_i^2\end{Bmatrix}=\begin{bmatrix}N & \sum_{i=1}^{n}\Omega_i^2\\ \sum_{i=1}^{n}\Omega_i^2 & \sum_{i=1}^{n}\left(\Omega_i^2\right)^2\end{bmatrix}\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix} \tag{12}$$

α and β can therefore be calculated using the following equation;

$$\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix}=2\begin{bmatrix}N & \sum_{i=1}^{n}\Omega_i^2\\ \sum_{i=1}^{n}\Omega_i^2 & \sum_{i=1}^{n}\left(\Omega_i^2\right)^2\end{bmatrix}^{-1}\begin{Bmatrix}\sum_{i=1}^{n}\sigma_i\\ \sum_{i=1}^{n}\sigma_i\Omega_i^2\end{Bmatrix} \tag{13}$$

Equation 13 is the desired relationship for calculating the proportional damping matrix coefficients from EMA frequency and damping estimates. α and β can then be used to add proportional damping to an FEA model.

If the damped FEA model is then solved for its modes, the following question arises: **How well do the modal frequencies and damping of the damped FEA model match the experimental modal parameters from which the damped model was derived?** This question is addressed in the following examples.

Beam Structure

We will consider the modes of the beam structure shown in Figure 1. This beam consists of three aluminum plates fastened together with cap screws. The top plate is fastened to the back plate with three screws, and the bottom plate is also fastened to the back plate with three screws.

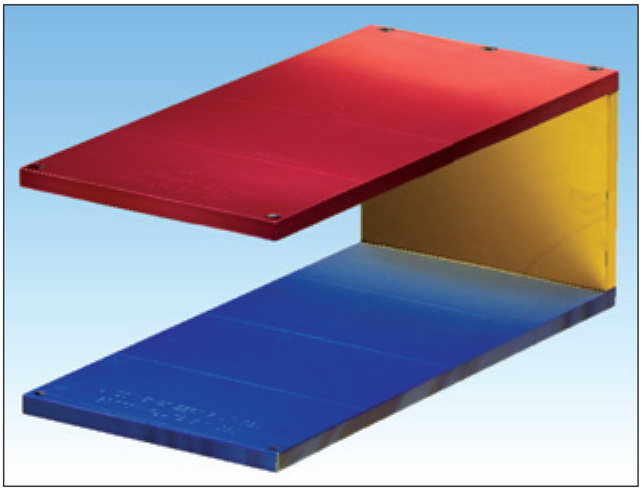

Figure 1. Jim Beam structure.

Modal Frequency & Damping

The modal frequency and damping for the first 11 (lowest frequency) FEA and EMA modes of the beam are listed in Table 1. The FEA frequencies were obtained as the eigenvalues of an undamped FEA model.[1] Because the FEA model was undamped, the FEA modes have no modal damping. The EMA parameters were estimated by curve fitting a set of experimentally derived FRFs.

Notice that each FEA modal frequency is less than its corresponding EMA frequency, indicating that the FEA model was less stiff than the actual beam. However, the modal assurance criterion (MAC) values between the FEA and EMA mode shape pairs indicate that most pairs are similar. (Two mode shapes are strongly correlated if their MAC value is 0.90 or greater.)

Two Extreme Cases

Two extreme cases are possible with the coefficients α and β; namely, $\alpha > 0$, $\beta = 0$ and $\alpha = 0$, $\beta > 0$.

Case 1 (β = 0). If $\beta = 0$, then viscous damping is only proportional to the mass distribution of the model, and Eq. 8 reduces to:

$$\alpha=2\sigma \tag{14}$$

If the proportional damping matrix coefficients are; $\alpha = 2\pi$, $\beta = 0$, then Eq. 14 states that all modes of the beam will have the same modal damping: $\sigma = \pi$ rad/sec = 0.5 Hz.

The coefficients $\alpha = 2\pi$, $\beta = 0$ were used to create a proportional damping matrix, and the damped FEA model was solved for its modes. The expected result (all modes with damping = 0.5 Hz), is shown in Table 2.

Case 2 (α = 0). If $\alpha = 0$, then viscous damping is only proportional to the stiffness distribution, and Eq. 8 reduces to:

$$\beta\Omega=\frac{2\sigma}{\Omega}=2\zeta \tag{15}$$

Table 1. Undamped FEA and damped EMA modes.

Mode	FEA Freq, Hz	FEA Freq, Hz	EMA Freq., Hz	EMA Damp.,Hz	Mode Shape MAC
1	61.405	0.0	96.944	5.6347	0.74
2	143.81	0.0	164.95	3.1125	0.96
3	203.71	0.0	224.57	6.5223	0.96
4	310.62	0.0	347.56	5.1552	0.95
5	414.4	0.0	460.59	11.502	0.93
6	442.6	0.0	492.82	4.6424	0.96
7	583.44	0.0	635.18	14.247	0.94
8	1002.2	0.0	1108.2	4.964	0.90
9	1090.8	0.0	1210.5	7.1292	0.88
10	1168.3	0.0	1322.6	7.2498	0.84
11	1388.2	0.0	1555.1	17.112	0.84

where $\zeta = \sigma/\Omega$ is the percent of critical damping of a mode. Eq. 15 indicates that for a given value of β, the percent of critical damping of each mode is proportional to its undamped frequency. For $\zeta = 1\%$ for the first mode (with frequency 61.4 Hz), Eq. 15 gives $\beta = 0.0000518$.

The coefficients $\alpha = 0$, $\beta = 0.0000518$ were used to create a proportional damping matrix, and the damped FEA model was solved for its modes. The results are shown in Table 3. Notice that the 61.4 Hz mode has the expected 1% damping, and also that the percent of critical damping increases as the modal frequency of the other modes increases.

Using EMA Frequency and Damping

The list of EMA damping values in Table 1 shows a wide range of values, between 3.11 and 17.11 Hz. Clearly, neither of the two extreme proportional damping cases exists in the actual beam structure: $\beta = 0$, which gives modes with the same modal damping, or $\alpha = 0$, which give modes with percent of critical damping that increases with increasing frequency.

The EMA frequency and damping estimates for the 11 modes in Table 1 were used to calculate α and β using Eq. 13. The least-squared-error estimates of α and β are: $\alpha = 76.6972$, $\beta = 8.0835 \times 10^{-7}$.

These coefficients were used to create a proportional damping matrix using the mass and stiffness matrices of the FEA model. This damped FEA model was then solved for its modes, and its modal frequency and damping values are listed in Table 4.

The modal damping values of the damped FEA model do exhibit monotonically increasing values with frequency, indicating a stronger proportionality to the stiffness than to the mass. This is similar to extreme Case 2. Even though the FEA damping values don't match the EMA damping values on a mode for mode basis, they are in the range of the EMA values. Nevertheless, this is still desirable for making the FEA model more useful for modeling the dynamics of the real structure.

Using FEA Frequency and EMA Damping

To determine the influence of modal frequency on the α and β values, the FEA frequencies were used instead of the EMA frequencies to calculate α and β. For this case, the least-squared-error estimates of α and β were: $\alpha = 76.4183$, $\beta = 1.0202 \times 10^{-6}$.

These estimates were then used to create a proportional damping matrix from the mass and stiffness matrices of the FEA model. The modal parameters of this damped FEA model are compared with the EMA parameters of the beam in Table 5.

Again, the FEA modal damping values are monotonically increasing with frequency, indicating a stronger proportionality to the stiffness than to the mass. These FEA damping values are closer to the EMA values than when the EMA frequencies were used, but there is no significant difference between the two solutions. The mode shape MAC values for this case are identical to the case where the EMA frequencies were used.

Conclusions

An equation was derived for calculating the proportional damping matrix coefficients (α and β) from two or more experimental modal frequency amd damping estimates. The equation for calculating α and β was derived as a least-squared error solution to an over-specified set of equations.

The modal damping values of two differently damped FEA models were compared with the EMA damping estimates for the

Table 2. Proportionally damped FEA modes ($\alpha = 2\pi$, $\beta = 0$).

Mode	Un-damped FEA Freq, Hz	Un-damped FEA Damp, Hz	Damped FEA Freq., Hz	Damped FEA Damp., Hz	Mode Shape MAC
1	61.405	0.0	61.403	0.49975	1.00
2	143.81	0.0	143.81	0.49975	1.00
3	203.71	0.0	203.71	0.49975	1.00
4	310.62	0.0	310.62	0.49975	1.00
5	414.4	0.0	414.4	0.49975	1.00
6	442.6	0.0	442.6	0.49975	1.00
7	583.44	0.0	583.44	0.49975	1.00
8	1002.2	0.0	1002.2	0.49975	1.00
9	1090.8	0.0	1090.8	0.49975	1.00
10	1168.3	0.0	1168.3	0.49975	1.00
11	1388.2	0.0	1388.2	0.49975	1.00

Table 3. Damped FEA modes ($\alpha = 0$, $\beta = 0.0000518$).

Mode	Un-damped FEA Freq, Hz	Un-damped FEA Damp, Hz	Damped FEA Freq., Hz	Damped FEA Damp., Hz	Damped FEA Damp., %
1	61.405	0.0	61.402	0.61361	0.99928
2	143.81	0.0	143.77	3.3657	2.3403
3	203.71	0.0	203.6	6.7533	3.3151
4	310.62	0.0	310.23	15.702	5.0549
5	414.4	0.0	413.46	27.946	6.7437
6	442.6	0.0	441.45	31.879	7.2026
7	583.44	0.0	580.81	55.396	9.4947
8	1002.2	0.0	988.81	163.46	16.31
9	1090.8	0.0	1073.5	193.63	17.751
10	1168.3	0.0	1147	222.11	19.012
11	1388.2	0.0	1352.3	313.61	22.591

Table 4. Damped FEA modes ($\alpha = 76.6972$, $\beta = 8.0835$ e-7).

Mode	Damped FEA Freq, Hz	Damped FEA Damp, Hz	EMA Freq., Hz	EMA Damp., Hz	Mode Shape MAC
1	61.1	6.1129	96.944	5.6347	0.73
2	143.68	6.1559	164.95	3.1125	0.97
3	203.62	6.2088	224.57	6.5223	0.96
4	310.56	6.3484	347.56	5.1552	0.96
5	414.35	6.5395	460.59	11.502	0.93
6	442.55	6.6008	492.82	4.6424	0.96
7	583.4	6.9678	635.18	14.247	0.94
8	1002.2	8.6542	1108.2	4.964	0.92
9	1090.8	9.125	1210.5	7.1292	0.90
10	1168.2	9.5695	1322.6	7.2498	0.86
11	1388.2	10.997	1555.1	17.112	0.84

Table 5. Damped FEA modes ($\alpha = 76.4183$, $\beta = 1.0202$ e-6).

Mode	Damped FEA Freq, Hz	Damped FEA Damp, Hz	EMA Freq., Hz	EMA Damp., Hz	Mode Shape MAC
1	61.102	6.0933	96.944	5.6347	0.73
2	143.68	6.1475	164.95	3.1125	0.97
3	203.62	6.2142	224.57	6.5223	0.96
4	310.56	6.3904	347.56	5.1552	0.96
5	414.35	6.6316	460.59	11.502	0.93
6	442.55	6.7091	492.82	4.6424	0.96
7	583.4	7.1723	635.18	14.247	0.94
8	1002.2	9.3007	1108.2	4.964	0.92
9	1090.8	9.8949	1210.5	7.1292	0.90
10	1168.2	10.456	1322.6	7.2498	0.86
11	1388.2	12.258	1555.1	17.112	0.84

structure. Although the FEA modal damping didn't match well on a mode-by-mode basis with all of the EMA estimates, the damping values of matching mode pairs were similar in value.

The reason for the disparity in the FEA versus EMA modal damping values is that proportional damping is restricted by the distribution of mass and stiffness in a structure, while its real-world damping mechanisms are not. In the case of the beam structure used here, it is quite clear that other significant damping mechanisms were present during the modal test, which dissipated energy that was not accounted for by the proportional damping model.

In fact, the beam structure was tested while resting on a foam rubber pad. The pad clearly had greater damping influence on the lower plate of the beam than on the other two plates. Therefore, depending on the mode shape components for the lower plate, some modes were more strongly influenced by the damping of the foam rubber base than others. This alone could account for the wide range of modal damping values (3.11 to 17.11 Hz).

From our example, one can conclude that a proportionally damped FEA model doesn't necessarily yield modes with damping that perfectly match experimental damping values. Nevertheless, this approach provides a straightforward way to add viscous damping to any FEA model and solve for modes that contain realistic modal damping values.

References

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