Techniques for Evaluation of Modal Vector Contamination

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Modal vectors frequently have small amounts of contamination or distortion from random errors or bias errors, particularly when compared to results from modeling where normal modal vectors are the common result since damping is not included in the model. To understand and possibly eliminate the contamination, tools are needed to evaluate the contamination. While the traditional modal assurance criterion (MAC) is useful, more sensitive methods are desirable. Several altered forms of MAC are reviewed for this purpose. These methods include evaluating the real part of a modal vector compared to the complex-valued modal vector (*rMAC*), evaluating the imaginary part of the modal vector compared to the complex valued modal vector (*iMAC*) and evaluating the real part of a modal vector compared to the imaginary part of the modal vector (*riMAC*). Weighted versions of each of these evaluations are also used (*rwMAC*, *iwMAC*,and *riwMAC*). These methods have shown to be very useful when evaluating modal vectors associated with close modal frequencies and suggest a need for improved processing (numerical estimation procedures for modal vectors) or "decontamination" (post-processing procedures for modal vector sets) are required.

The evolution of modal parameter estimation over the last 40 years or so has changed the way modal vectors are estimated from experimental data. The progression from single-measurement modal parameter estimation to autonomous (MIMO) modal parameter estimation has meant that the modal vector coefficients that once were estimated DOF by DOF and mode by mode can now be estimated vector by vector (including all DOFs) from clusters of estimates of each modal vector in MIMO procedures.

This has resulted in statistically significant estimations of the individual modal vectors that reduce the impact of measurement noise as well as other random and bias errors. In the end, the modal vectors always have some small amount of contamination. When a structure is tested where normal modes are expected, the estimated modal vectors will always contain a small amount of contamination that will yield a slightly complex estimate of the modal vectors.

For this situation, the contamination can often be ignored or eliminated through a real normalization procedure. This can be justified, particularly when the contamination appears to be dominantly random. However, when the contamination is biased, this justification becomes complicated. Even with the most sophisticated modal parameter estimation algorithms and numerical procedures, the contamination will often be biased in the form of contamination that looks like a nearby mode. This indicates that the estimated modal vectors satisfy whatever algorithm and numerical procedure are being utilized, but the estimated modal vectors still contain characteristics that may be perceived as a nonphysical result.

Recent use of autonomous modal parameter estimation methods indicates that these small amounts of contamination still persist even when statistically significant data are included in the estimation of the modal vectors, and estimation of the modal vectors involves alternate numerical methods. The common form of this contamination is most notable when the modal frequencies are closely spaced or repeated in frequency. In these cases, when the modal vectors are expected to be real-valued, normal modes, the estimated modal vectors will often contain a small imaginary valued component that correlates with the dominant (real-valued) characteristic of a nearby modal vector.

Nomenclature

N_i = Number of inputs
N_0 = Number of outputs
N_S = Short dimension size
N_L = Long dimension size
λ_r = Complex modal frequency (rad/sec)
$\lambda_r = \sigma_r + j\omega_r$
σ_r = Modal damping
ω_r = Damped natural frequency
$\{\psi_r\}$ = Base vector (modal vector)
$\{\phi_r\}$ = Pole-weighted base vector (state vector)
r = Mode number
MAC = Modal assurance criterion
rMAC = MAC (real part versus complex)
iMAC = MAC (imaginary part versus complex)
riMAC = MAC (real part versus imaginary part)
wMAC = Weighted modal assurance criterion
rwMAC = Weighted MAC (real part versus complex)
iwMAC = Weighted MAC (imaginary part versus complex)
riwMAC = Weighted MAC (real part versus imaginary part)

Autonomous Modal Parameter Estimation

Some comments about how modal vectors are estimated when using autonomous modal parameter estimation methods are in order.[1-3] In the end, the modal vector contamination that is being studied is present in all modal parameter estimation approaches. However, the autonomous modal parameter estimation procedures often use a statistically based solution that involves a singular-value decomposition of a cluster of modal vectors estimates. This yields an extremely good result, where the modal vectors have much less contamination than that found historically. Even so, the modal vector contamination problem cannot be eliminated.

The following discussion is a brief summary of how the common statistical subspace autonomous modal identification (CSSAMI) method estimates the modal vectors. Essentially, any modal parameter estimation algorithm can be utilized to get a consistency diagram. This consistency diagram represents hundreds of solutions for the possible modal parameters (modal frequencies and modal vectors). The vectors in these solutions are combined with the modal frequencies to create state vectors. Now the hundreds of state vectors can be sorted into clusters, where each cluster represents a single modal vector.

This sorting procedure involves the modal assurance criterion between all of the state vectors. The final modal frequency and modal vector can now be determined from the singular value decomposition of each cluster. This is a slightly different procedure than historical methods that used least-squares or weighted least-squares methods to determine modal vectors via a partial fraction residue model.

Note that much of the background of the CSSAMI method is based upon the unified matrix polynomial algorithm (UMPA).[4-5] This means that this autonomous method can be applied to both low- and high-order modal parameter estimation methods with short or long dimension modal (base) vectors. These different methods can now be combined in one procedure.

In these cases, it may be useful to solve for the complete unscaled or scaled modal vector of the large dimension N_L. This will extend the temporal-spatial information in the modal (base) vector so that the vector will be more sensitive to change. This characteristic is what gives the CSSAMI autonomous method a robust ability to distinguish between computational and structural modal parameters. (Please refer to a series of previous papers in order to get an

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overview of the methodology and to view application results for several cases.[6-8])

Modal Assurance Criterion

The traditional modal assurance criterion (MAC) computation[9-12], restated in Equation 1, is widely used in modal parameter estimation and structural dynamics to sort the numerous possible solutions of modal vectors from either modeling or experiment:

$$MAC_{cd} = \frac{\left|\{\psi_c\}^H\{\psi_d\}\right|^2}{\{\psi_c\}^H\{\psi_c\}\{\psi_d\}^H\{\psi_d\}} = \tag{1}$$

$$\frac{\{\psi_c\}^H\{\psi_d\}\{\psi_d\}^H\{\psi_c\}}{\{\psi_c\}^H\{\psi_c\}\{\psi_d\}^H\{\psi_d\}}$$

Once modal vectors are estimated in any modal parameter estimation procedure, the MAC computation is often used to evaluate the quality of the solutions. This begins with an evaluation of the MAC between all of the modal vectors in the final set to ascertain whether the modal set is an independent set of vectors. This often involves including the estimates of the modal vectors associated with the conjugate poles.

Since the conjugate poles and vectors are estimated separately, if nonconjugate relationships exist between the associated modal vector estimates (between the modal vector for pole and the modal vector of the conjugate pole), the MAC between these two vectors will not be unity as expected. A number of users have noted that this often correlates with modal vectors that are exhibiting some unexpected characteristics.

Figure 1 is a graphical representation of this situation. While the MAC values are acceptable, the comparisons between modal vectors and the associated conjugate modal vectors do exhibit slightly lowered consistency or correlation.

When the last three modal vectors are visualized, as in Figures 2 and 3, no particular problem can be noted until the modal vectors are animated. Then, the two modes in Figure 2 clearly show a small complex mode characteristic. Since these results are statistically consistent across many solutions, the limitations of the data, both in frequency and spatial resolution, are the root of the problem.

Unlike the historical approach to estimation of the modal vectors, many recent modal parameter estimation algorithms, including the autonomous procedures, are based on numerical processing methods like singular value decomposition (SVD). The solutions that are identified, based on the data associated with a cluster of estimates, have no physical or causal constraint. An example of a physical or causal constraint would be the expectation of real-valued, normal modes for systems where no expectation of nonproportional damping is likely. SVD methods will identify the most dominant unitary (orthogonal and unit length) vectors in a cluster, yielding a complex-valued vector in general. Experience has shown that when modes are very close in frequency with minimal spatial resolution, the complex-valued vectors will still show significant independence.

However, when these complex-valued vectors are examined closely, the nondominant portion of the complex-valued vector often correlates very highly with one or more nearby modal vectors. This can be examined by several variants of the MAC calculation and the weighted MAC calculation.

Weighted Modal Assurance Criterion

Identifying the potential contamination of modal vectors is helpful to the thorough understanding of modal parameter estimation algorithms and autonomous procedures as well as being instructive for potential removal of the contamination.[13] If some sort of real normalization is desirable (to match well with an undamped analytical model, for example), understanding the contamination that is being removed is a prerequisite to any procedure. Random contamination may simply be ignored, smoothed or averaged out, but if the contamination is related to nearby modes, it may indicate that the modal parameter estimation may need further evaluation or that more data are required.

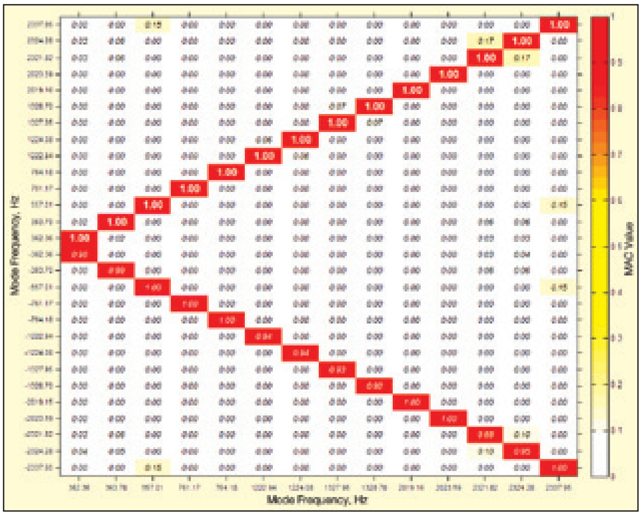

Figure 1. MAC of modal vectors and conjugate modal vectors.

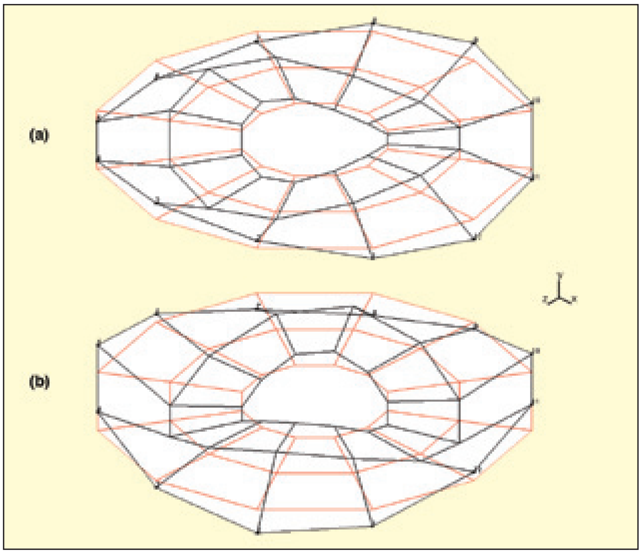

Figure 2. C-plate example: modal vectors – 2312.8 Hz (a) and 2324.3 Hz (b).

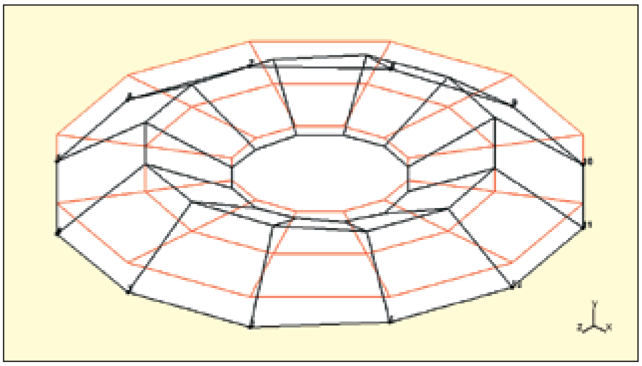

Figure 3. C-plate example: modal vector – 2337.9 Hz.

For this evaluation of the modal vector contamination, it is easier to first rotate each complex-valued modal vector to a real (or imaginary) dominant vector. This is done by using a least-squares method to identify the rotation of the modal vector away from the real or imaginary axis and then using the associated complex phasor to rotate each original complex-valued modal vector to a new complex-valued modal vector that aligns with the real or imaginary axis.[13] For all following discussions, the original complex-valued modal vectors are rotated to be dominantly real-valued. It is convenient, for display reasons to also normalize the new complex-valued modal vector to a unity maximum or unity vector length. Naturally, the rotation and rescaling must be considered in any final estimates of modal scaling (modal mass, modal A, residue, etc.)

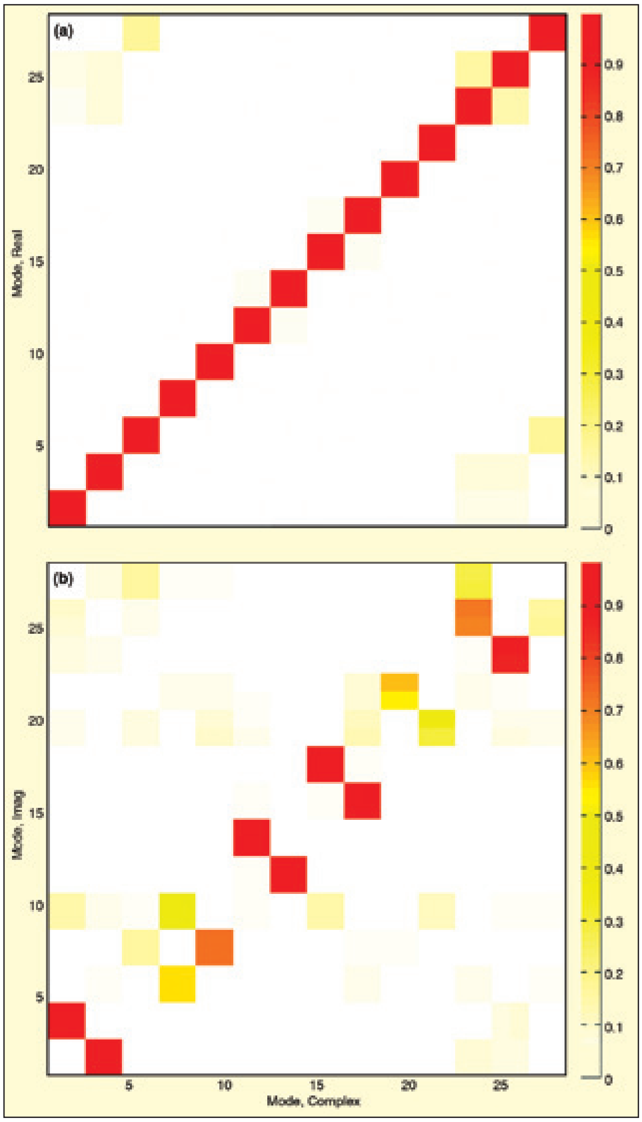

Figure 4. Real and imaginary MAC evaluations.

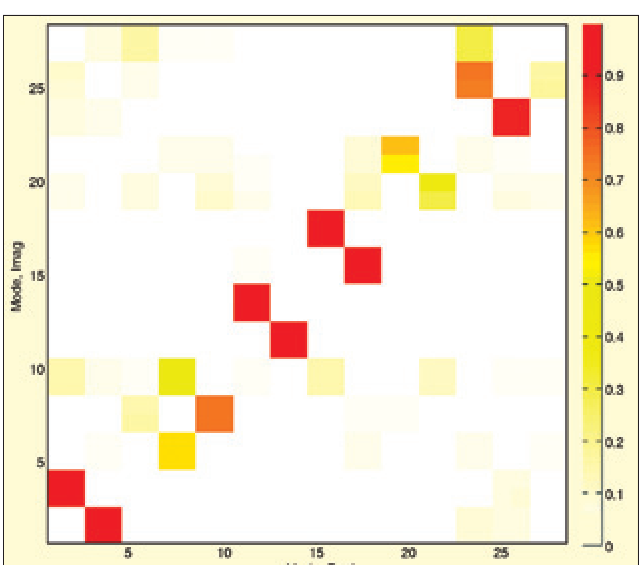

Figure 5. Real versus imaginary MAC evaluation.

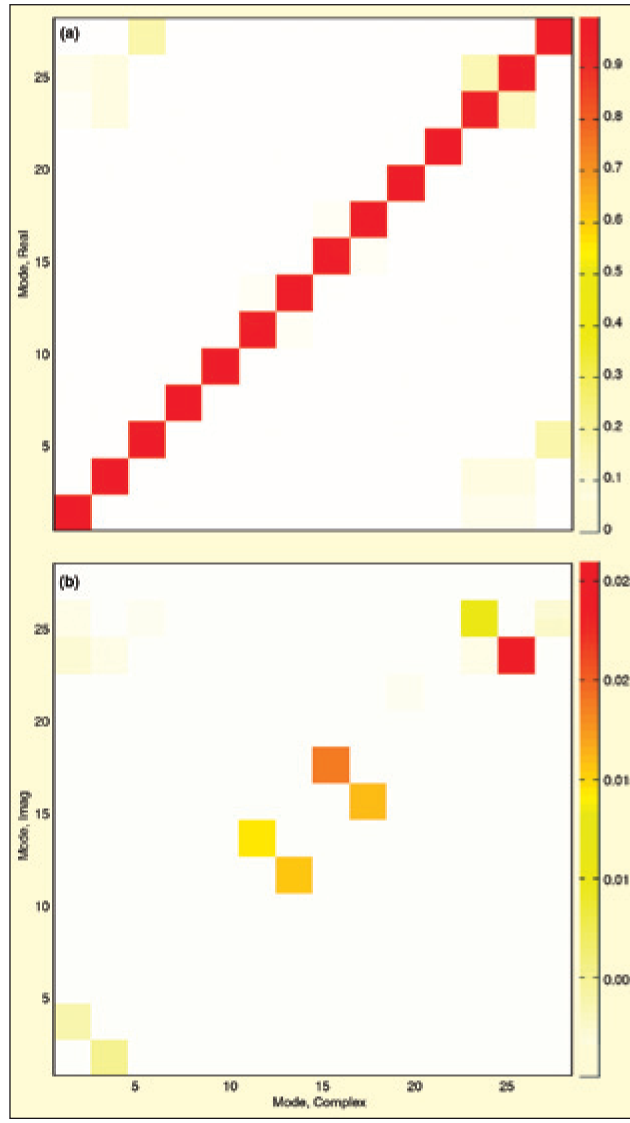

Figure 6. Real and imaginary weighted MAC evaluations.

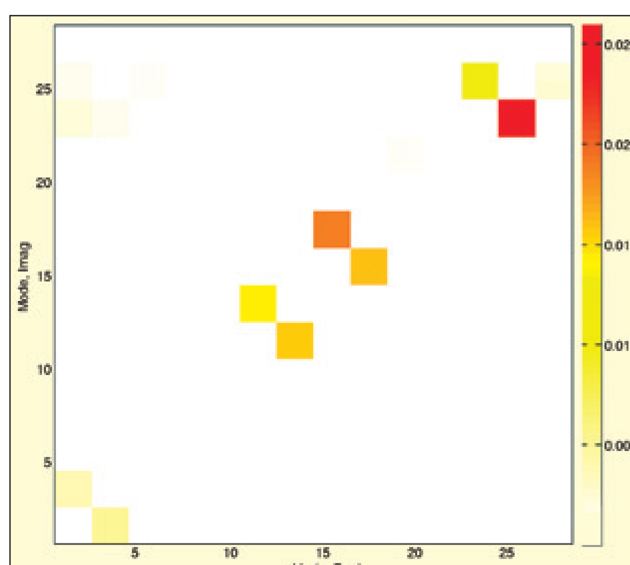

Figure 7. Real versus imaginary weighted MAC evaluation.

To understand the nature of the possible modal vector contamination in a complex-valued modal vector, three conventional MAC calculations can be performed (1) between the real parts of the modal vectors and the complex-valued modal vectors, *rMAC*, (2) between the imaginary parts of the modal vectors and the complex-valued modal vectors, *iMAC*, and (3) between the real parts of the modal vectors and the imaginary parts of the modal vectors, *riMAC*.

These three MAC calculations and the interpretation of these MAC values will be sensitive to the rotation and normalization of the complex-valued modal vector estimates. The following use and discussion assumes that the complex-valued modal vectors have been rotated so that the central axis of the complex-valued modal vector is centered on the real axis.

These three MAC computations identify: (1) that the real part of the modal vector is the dominant part of the complex-valued modal vector, *rMAC*; (2) that the imaginary part of the modal vector is the dominant part of the complex-valued modal vector, *iMAC*; and (3) that the real and imaginary parts of the modal vector are, or are not, related to one another.

All MAC computations in this case are, as always, bounded from zero to one. If near normal modes are expected: (1) the *rMAC* should be close to one, (2) the *iMAC* should be close to zero; and (3) the *riMAC* should also be close to zero. Note in the following definitions, complex-valued modal vectors c and d can again be any of the modal vectors that the user wishes to include in the evaluation:

$$rMAC_{cd} = \frac{\left(\mathrm{Re}\{\psi_c\}^H\right)\{\psi_d\} \quad \{\psi_d\}^H\left(\mathrm{Re}\{\psi_c\}\right)}{\left(\mathrm{Re}\{\psi_c\}^H\right)\left(\mathrm{Re}\{\psi_c\}\right) \quad \{\psi_d\}^H\{\psi_d\}} \tag{2}$$

$$iMAC_{cd} = \frac{\left(\mathrm{Im}\{\psi_c\}^H\right)\{\psi_d\} \quad \{\psi_d\}^H\left(\mathrm{Im}\{\psi_c\}\right)}{\left(\mathrm{Im}\{\psi_c\}^H\right)\left(\mathrm{Im}\{\psi_c\}\right) \quad \{\psi_d\}^H\{\psi_d\}} \tag{3}$$

$$riMAC_{cd} = \frac{\left(\mathrm{Re}\{\psi_c\}^H\right)\left(\mathrm{Im}\{\psi_d\}\right) \quad \left(\mathrm{Im}\{\psi_d\}^H\right)\left(\mathrm{Re}\{\psi_c\}\right)}{\left(\mathrm{Re}\{\psi_c\}^H\right)\left(\mathrm{Re}\{\psi_c\}\right) \quad \left(\mathrm{Im}\{\psi_d\}^H\right)\left(\mathrm{Im}\{\psi_d\}\right)} \tag{4}$$

Figures 4 and 5 are graphical representations of Equations 2 through 4. Each block or cluster in these diagrams contains the information from both the complex modal frequency and the associated conjugate modal frequency. The *rMAC* in Figure 4 shows that the modal vectors are real dominant and linearly independent. The *iMAC* in Figure 4 and the *riMAC* in Figure 5 both show that the imaginary portion of the vectors are linearly and strongly related to a nearby mode, which is frequently the pseudo-repeated root twin to the mode in this case.

These graphical representations indicate that the imaginary part (contamination) of a given mode is strongly related to the real (dominant) part of the modal vector associated with its pseudo-repeated root companion. This is consistent with theory that explains the cause of a complex-valued modal vector when two real-valued modal vectors are close in frequency and misidentified as a single modal vector.

The above MAC evaluations identify whether, and how, the contamination of a complex-valued modal vector is related to another of the identified modal vectors. However, the MAC computation is normalized by vector length, vector by vector, for the vectors used in the calculation. A weighted MAC can be used to determine the degree or scale of the contamination. The following three definitions of the weighting for each of the above MAC calculations limit the associated MAC value to a fraction of the zero-to-one scale. If near normal modes are expected, (1) the weighting and *rwMAC* should be close to one; (2) the weighting and *iwMAC* should be close to zero; and (3) the combined weighting and *riwMAC* should also be close to zero. Note that in the following definitions, complex-valued modal vectors c and d can again be any of the modal vectors that the user wishes to include in the evaluation:

$$rwMAC_{cd} = rW_c \times rMAC_{cd} \text{ where } rW_c = \frac{\left(\mathrm{Re}\{\psi_c\}^H\right)\left(\mathrm{Re}\{\psi_c\}\right)}{\{\psi_c\}^H\{\psi_c\}} \tag{5}$$

$$iwMAC_{cd} = iW_c \times iMAC_{cd} \text{ where } iW_c = \frac{\left(\mathrm{Im}\{\psi_c\}^H\right)\left(\mathrm{Im}\{\psi_c\}\right)}{\{\psi_c\}^H\{\psi_c\}} \tag{6}$$

$$riwMAC_{cd} = rW_c \times iW_d \times riMAC_{cd} \tag{7}$$

Figures 6 and 7 are graphical representations of Equations 5 through 7. These figures yield the same conclusions as Figures 4 and 5. In addition, the *iwMAC* and *riwMAC* values show that the contamination is at a relatively low level.

At this point, now that the contamination of the complex-valued modal vectors can be confirmed to be from the dominant portion (real part) of the other complex-valued modal vectors and that the contamination is not significant, a strategy for determining the best set of real-valued modal vectors can be identified. One reasonable option would be to place the real parts and imaginary parts of each complex-valued modal vector into a matrix as separate real-valued vectors.

A singular-value decomposition of this real-valued matrix will yield real-valued singular vectors, and the most significant singular vectors equal to the original number of complex-valued modal vectors associated with the largest singular values can be used as the final set of real-valued normal modes. A simpler solution would be to eliminate the imaginary parts, since the scale of the contamination is shown to be small.

Summary

With the advent of more computationally powerful computers and sufficient memory, it has become practical to evaluate sets of solutions involving hundreds or thousands of modal parameter estimates and to extract the common information from those sets. In many cases, autonomous procedures give very acceptable results, in some cases superior results, in a fraction of the time required for an experienced user to get the same result. The modal assurance criterion, both unweighted and weighted, is instrumental in evaluating the quality of the modal vector results.

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