Know Your Notches . . . Facts About Antiresonances

George Fox Lang, Associate Editor

If you study vibrations, your world has measurable peaks and valleys. Business, in general, has its ups and downs. Just as your accountant would rather discuss the economic highs of your business, your vibration mentor was more likely to tell you all about the properties of resonance peaks rather than describing the accompanying antiresonance valleys. This is unfortunate; those antiresonance depressions have a lot to tell us.

In electronic servomechanism parlance, an antiresonance is a system zero as opposed to a system pole (a resonance). To most mechanical engineers, an antiresonance is just a sharp amplitude dip between two resonance peaks seen in a log-log vibration response measurement. (Without the use of log amplitude scaling, we would be totally unaware of these minima.) The electrical engineer and his preferred log-scaled plot has given us something quite important; he has given us the key to understanding how to correct a vibration problem.

Consider a simple two-degree-of-freedom (2-DOF) lumped-mass system as shown in Figure 1. Let's apply measured forces to each mass, in turn, and monitor the response acceleration of the two masses. The resulting frequency response function (FRF) magnitudes are plotted in Figure 2 with the frequency axis normalized by dividing frequency (in Hertz) by $1/2\pi\sqrt{K/M}$. Note that Figure 2 actually shows *four* traces; structural reciprocity causes the two *spatial transfer* FRFs, $\ddot{X}_1/F_2$ *and* $\ddot{X}_2/F_1$ to be identical. Observe that all four traces exhibit sharp peaks at the two resonance frequencies.

The two *driving-point* FRFs, $\ddot{X}_1/F_1$ *and* $\ddot{X}_2/F_2$ each exhibit a sharp notch between the two resonances. These are antiresonances. While the resonance frequencies (the natural frequencies) are spatially invariant, antiresonance frequencies vary with the DOF from which they are detected. A driving-point FRF will always have an antiresonance between each pair of resonances. Spatial transfer measurements may or may not exhibit antiresonances. In

our simple 2-DOF example, no antiresonance notch is evident in $\ddot{X}_1/F_2$ *and* $\ddot{X}_2/F_1$, because of the two *mode shapes* exhibited by our structure.

At a normalized frequency of 0.62, the first resonance has both masses moving in phase with one another. Both springs experience tension simultaneously as the masses move upward. They compress simultaneously as the masses descend. At a normalized frequency of 1.62, the second mode shape has the masses moving in phase opposition. When the lower mass rises, the upper mass descends. This places the lower spring in tension and the upper one in compression. All of these conditions reverse when the lower mass moves downward past the equilibrium position. The (dotted) spatial transfer FRFs reflect these shapes. The lower-frequency spike has a positive sign, while the upper one is negative. In modal analy-

sis parlance, the lower resonance has a positive *residue*, while the upper one has a negative residue.

In the above log-log acceleration FRF plots, each resonance frequency marks the intersection of a rising (+2 slope) "spring line" or low-frequency *stiffness-reciprocal asymptote* with a (flat) *mass-reciprocal asymptote* above it. These lines are asymptotes of opposite phase. At their intersection, they cancel one another, yielding a sharp spike, whose amplitude is controlled solely by the damping in the system (0.5% in this example).

The antresonance frequencies represent the intersection of a mass-reciprocal line extending above a resonance with a rising stiffness-reciprocal asymptote leading to the next resonance. If the two bounding resonances are of the same phase, the intersection results in a notch of near-zero amplitude. If they are in phase opposition, no cancellation occurs. Instead, a gentle rounded valley results. When a driving-point FRF is measured, all of the
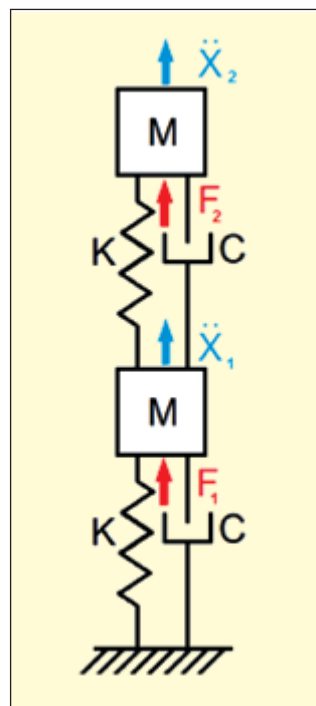

Figure 2. Four acceleration/force FRFs.
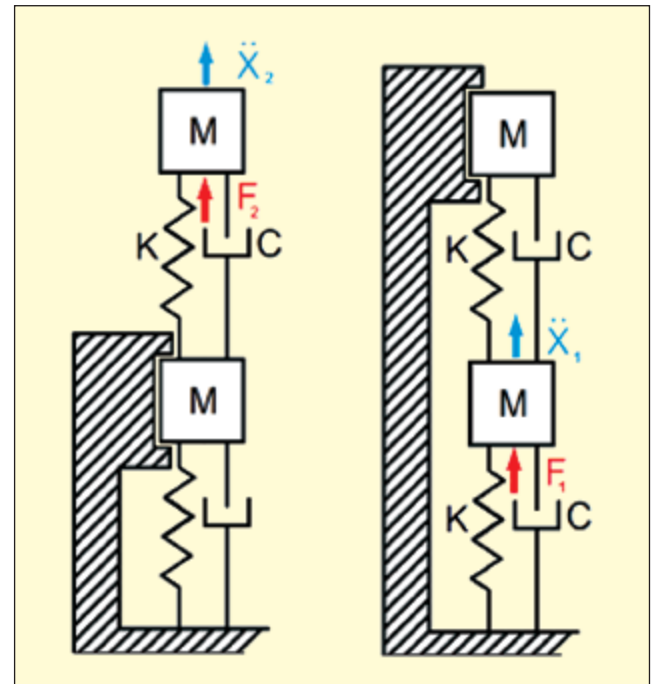
Figure 3. Applying constraints to the two DOFs.

Figure 1. Base-constrained 2-DOF lumped-mass system.
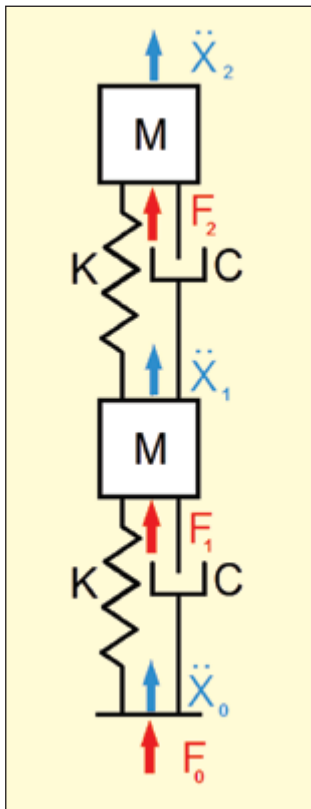
Figure 4. An "ignorable" coordinate shake more commonly, a base shake.

modes are represented with the same phase; all the resonances have positive residues and strong antiresonance notches result. In a spatial-transfer FRF, adjacent modes may or may not be in phase; if they are, a notch results. Should adjacent residues in a spatial-transfer FRF be of opposite sign, there will be no antiresonance notch between them.

If we measure a series of driving-point FRFs around a structure, we will find the antiresonance frequencies change from location to location. Each antiresonance is bounded by a pair of resonances, whose natural frequencies are constant around the structure. A *node* is a place (and direction) on the structure where an antiresonance frequency is equal to one of its bounding resonance frequencies. The result is that this DOF is zero valued in the intersected resonance's shape.

Predicting Change Effects

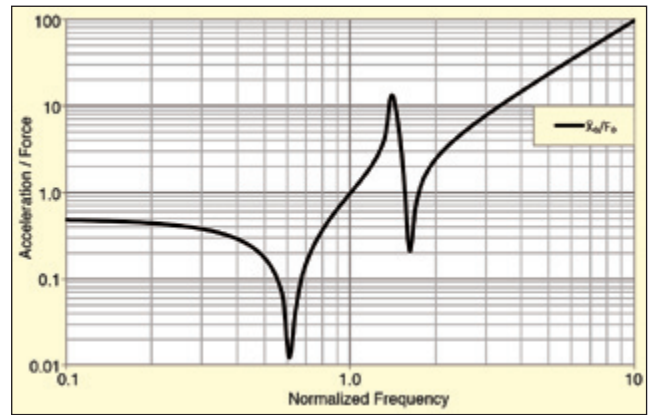*Antiresonances also represent the structure's resonances under different boundary conditions*. To be specific, if a DOF is "grounded" so that it may not move, the resulting constrained structure will have resonance frequencies equal to the antiresonance frequencies of the unconstrained structure. This provides a wonderful predictive tool for modification analysis. By way of example, consider the effects of constraining our simple 2-DOF structure as shown in Figure 3.

First, we ground the lower mass, forcing the constraint $\ddot{X}_1 = \dot{X}_1 = X_1 = 0$. This results in a single-degree-of-freedom model with an undamped natural frequency equal to $1/2\pi\sqrt{K/M}$ or a normalized frequency of 1. This is exactly the antiresonance frequency shown by the red $\ddot{X}_1/F_1$ trace.

Second, constrain the upper mass, forcing $\ddot{X}_2 = \dot{X}_2 = X_2 = 0$ and a single-DOF system results. However, this system has twice the restoring force (2K) and therefore an undamped natural frequency of $1/2\pi\sqrt{2K/M}$ or a normalized frequency of $\sqrt{2}$. This agrees exactly with the antiresonance frequency of the black $\ddot{X}_2/F_2$ trace.

Don't ignore the ignorable

Now, what might happen if we pick our test structure up and bolt it down on a shaker table instead of to a rigid foundation? In essence, we are changing the systems boundary conditions from base-constrained to free-free. Presuming we measure the force, F_0, applied by the shaker to the bottom of the lower spring and damper as shown in Figure 4, we can form another driving-point FRF, $\ddot{X}_0/F_0$. The result is plotted in Figure 5.

Clearly, the form of this FRF is very different from the base-constrained, driving-point measurements of Figure 2, yet they are strongly related. The most important fact is that the two antiresonances seen are at exactly the 0.62 and 1.62 normalized resonance frequencies of the base-constrained configuration. The single resonance between these antiresonances is a genuine free-free mode. Looking at these actions in reverse sequence, grounding the base DOF of this free-free structure results in a constrained structure with resonance frequencies matching the two antiresonances of the free-free shake.

Less obvious is the fact that the first singularity encountered in a free-free shake is always an antiresonance notch, not a resonance spike. In contrast, constrained structure tests always exhibit FRFs that have a resonance lower in frequency than the first antireso-

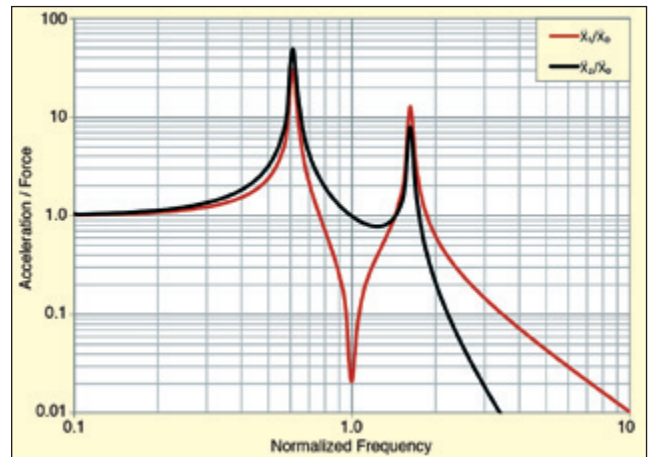Figure 5. Acceleration/force FRF from a base shake.

Figure 6. Transmissability FRFs measured in a base shake.
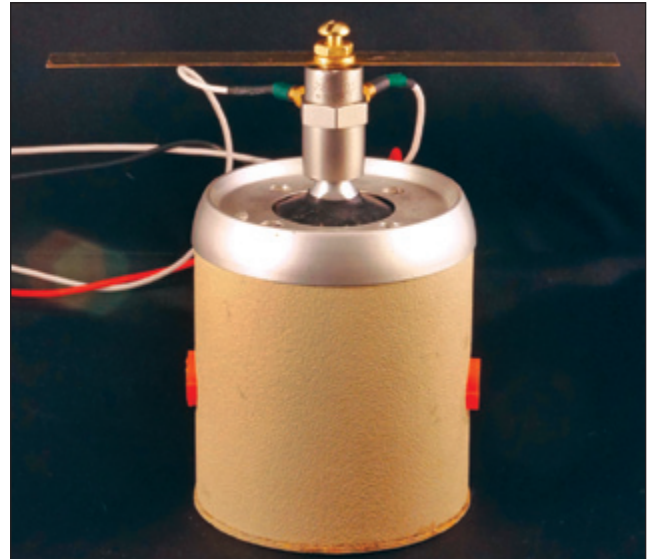

Figure 7. Base shake of center-clamped beam using PCB 288M05 impedance head and LDS V203 shaker.

nance. Alternatively, one can recognize that the first resonance encountered in a free-free shake is actually a rigid body translation of the entire system occurring at zero Hertz. It is also interesting to note that the mass-reciprocal asymptote at low frequency "weighs" the entire structure by exhibiting a normalized $\ddot{X}_0/F_0$ magnitude of 1/2M.

Modal analysts term tests of this geometric form an *ignorable coordinate* shake, indicating that a coordinate normally constrained to ground is the forcing degree of freedom. Others will recognize this configuration simply as an environmental shake test, normally used to prove robustness rather than to specifically identify resonances. However, the above FRF is rarely measured in an environ-

Derivation of Equations
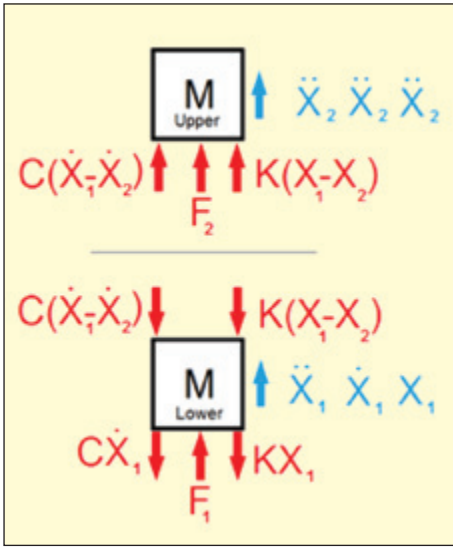
Free-body diagrams of the two masses show the forces acting upon each. Summing the forces and equating them to the mass-acceleration products yields the familiar matrix force balance statement:

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}\begin{Bmatrix} \ddot{X}_2 \\ \ddot{X}_1 \end{Bmatrix} + \begin{bmatrix} C & -C \\ -C & 2C \end{bmatrix}\begin{Bmatrix} \dot{X}_2 \\ \dot{X}_1 \end{Bmatrix} + \begin{bmatrix} K & -K \\ -K & 2K \end{bmatrix}\begin{Bmatrix} X_2 \\ X_1 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_1 \end{Bmatrix} \tag{1}$$

Applying the LaPlace Transform (with all initial conditions equal to zero) allows the equations to be written more compactly:

$$\begin{bmatrix} (MS^2 + CS + K) & -(CS + K) \\ -(CS + K) & (MS^2 + 2CS + 2K) \end{bmatrix}\begin{Bmatrix} X_2 \\ X_1 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_1 \end{Bmatrix} \tag{2}$$

Inverting the matrix of Equation 2 provides the basic FRF solutions for forced response. Applying synthetic double-differentiation allows these to be stated in terms of acceleration:

$$\begin{Bmatrix} \ddot{X}_2 \\ \ddot{X}_1 \end{Bmatrix} = \frac{S^2}{(MS^2+CS+K)(MS^2+2CS+2K)-(CS+K)^2}\begin{bmatrix} (MS^2+2CS+2K) & (CS+K) \\ (CS+K) & (MS^2+2CS+2K) \end{bmatrix}\begin{Bmatrix} F_2 \\ F_1 \end{Bmatrix} \tag{3}$$

Evaluating Equation 3 for $S = j2\omega f$ provides the FRFs of the base-constrained system. Note the denominator term is the system's characteristic equation, and its roots are the damped natural frequencies and damping in radians per second.

When the system is placed on a shaker and subjected to a base shake, the equations become homogeneous but non-square:

$$\begin{bmatrix} (MS^2+CS+K) & -(CS+K) & 0 \\ -(CS+K) & (MS^2+2CS+2K) & -(CS+K) \end{bmatrix}\begin{Bmatrix} X_2 \\ X_1 \\ X_0 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{4}$$

These can be restated as a square set of non-homogeneous equations by employing the result (Equation 3) as:

$$\begin{Bmatrix} \ddot{X}_2 \\ \ddot{X}_1 \end{Bmatrix} = \frac{S^2}{(MS^2+CS+K)(MS^2+2CS+2K)-(CS+K)^2}\begin{bmatrix} (MS^2+2CS+2K) & (CS+K) \\ (CS+K) & (MS^2+2CS+2K) \end{bmatrix}\begin{Bmatrix} 0 \\ (CS+K)X_0 \end{Bmatrix} \tag{5}$$

This permits the transmissabilities to be written explicitly:

$$\begin{Bmatrix} \ddot{X}_2 / \ddot{X}_0 \\ \ddot{X}_1 / \ddot{X}_0 \end{Bmatrix} = \frac{(CS+K)}{(MS^2+CS+K)(MS^2+2CS+2K)-(CS+K)^2}\begin{Bmatrix} (CS+K) \\ (MS^2+2CS+2K) \end{Bmatrix} \tag{6}$$

The force applied by the shaker to the base of the system can be identified as:

$$F_0 = (CS+K)(X_0 - X_1) \tag{7}$$

The motion, X_1, is given by (Equation 6). Specifically:

$$X_1 = \frac{(CS+K)(MS^2+CS+K)}{(MS^2+CS+K)(MS^2+2CS+2K)-(CS+K)^2}X_0 \tag{8}$$

So the acceleration/force FRF for the ignorable coordinate X_0 may be written:

$$\ddot{X}_0 / F_0 = \frac{S^2\left[(MS^2+CS+K)(MS^2+2CS+2K)-(CS+K)^2\right]}{(CS+K)\left[(MS^2+CS+K)(MS^2+2CS+2K)-(CS+K)^2-(CS+K)(MS^2+CS+K)\right]} \tag{9}$$

mental shake, since the force applied to the test article is rarely measured. Instead, motional patterns at resonances are identified from *transmissibility* FRFs. Transmissabilites are measured by using the shaker table's acceleration ($\ddot{X}_0$) as an FRF reference.

Note from the two transmissibility plots, $\ddot{X}_1 / \ddot{X}_0$ and $\ddot{X}_2 / \ddot{X}_0$ shown in Figure 6, that the 0.62 and 1.62 normalized frequency resonances of the base-constrained structure present as sharp peaks. It is interesting to note that exactly the same plot results, whether the motions are measured as acceleration, velocity or displacement. Further, the same transmissibility FRF can be found by applying forces to a base-constrained structure and measuring the resulting reaction force, F_0. That is $\ddot{X}_n / \ddot{X}_0 = \dot{X}_n / \dot{X}_0 = X_n / X_0 = F_0 / F_n$!

Some Experimental Confirmation

Now it is time to demonstrate that these mathematical exercises actually model observable physics. Figure 7 shows a simple 6-inch brass beam center-mounted to an impedance head on a small electrodynamic shaker. The impedance head provides 1000 mv/lb and 100 mV/g force and acceleration signals at the driving point of the structure; that is, it measures $\ddot{X}_0 / F_0$. An (uncontrolled) base shake was conducted using white random noise over a 250-Hz span. Figure 8 presents the results over 10 to 100 Hertz. Compare this figure with Figure 5.

In Figure 8, the first singularity is clearly an antiresonance; this occurs at 42.9688 Hz. A resonance peak occurs at 50.4688 Hz.
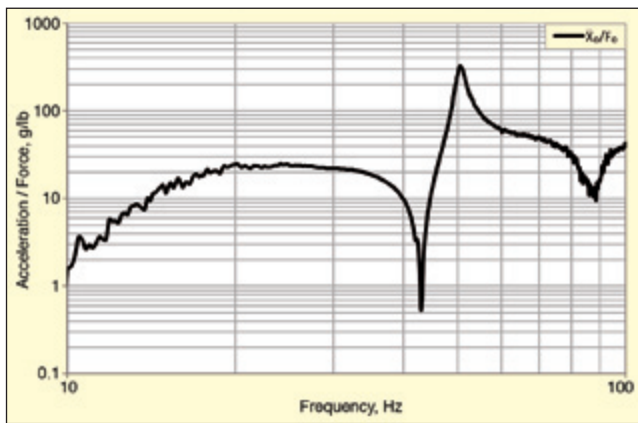
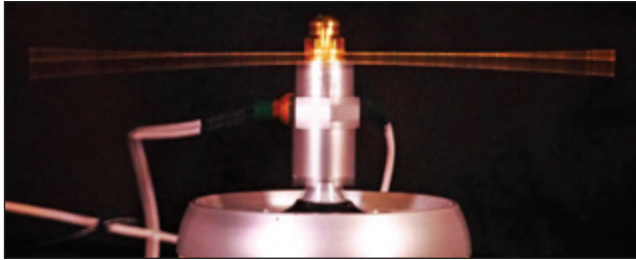Figure 8. Measured $\ddot{X}_0 / F_0$ of base-driven, center-clamped beam.

Figure 9. Rigid-body motion below first antiresonance frequency; response shape at 30 Hz shown.
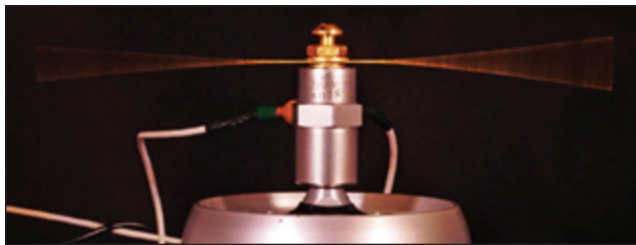

Figure 10. Sinusoidal response shape at 42.9688 Hz antiresonance; note stationary shaker and beam center.
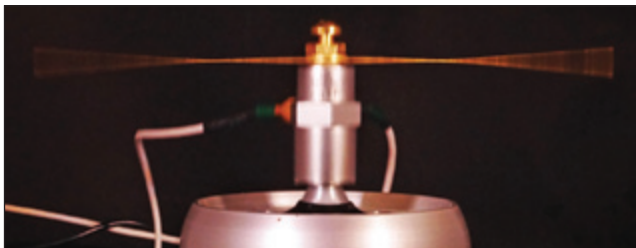

Figure 11. Sinusoidal response shape at 50.4688 Hz resonance; note shaker motion at beam center.

Below the first antiresonance, the $\ddot{X}_0 / F_0$ measurement is essentially flat, and it "weighs" the entire test structure (with slightly less than 1/20 lb indicated). The actual beam and its mounting hardware weigh 21.57 grams or 1/21.07 of a lb. Below 10 Hz, the measured data are compromised by the AC coupling of the sensor power supplies and the analyzer. This probably causes the gentle roll-off below 20 Hz seen here.

The operating deflection shapes (ODS) associated with this measurement are interesting to view. Below the first antiresonance, the beam structure acts like a rigid body. Figure 9 illustrates the typical response shape when the shaker is driven with a 30-Hz sine wave. In this configuration, the impedance head experiences the full mass of the DUT. As the antiresonance is approached, the apparent mass of the DUT increases. At each antiresonance, the detected mass is a local maximum. In contrast, each free-free resonance provides a local minimum effective mass.

Figure 10 shows the deformation pattern when the beam is driven at the antiresonance frequency with a 42.9688 sinewave. Note each
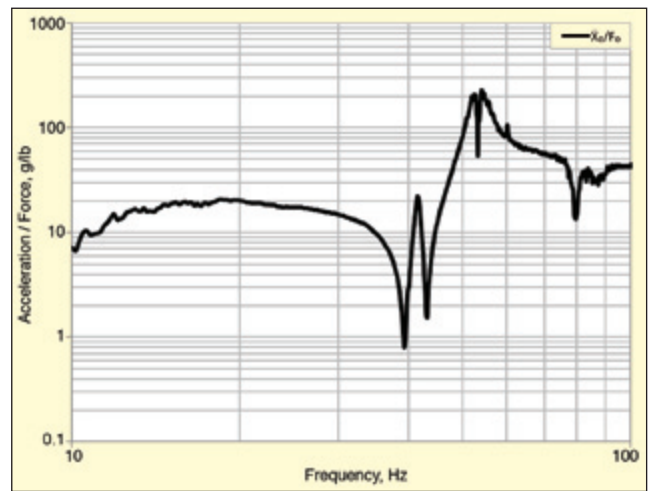

Figure 12. Repeat of Figure 10 with beam slightly off center, illustrating "modal splits."

limb of the beam responds with a shape identical to a cantilever's first mode shape. That is, the driving shaker is (essentially) standing still, while the beam tips exhibit maximum motion.

In contrast, when the sine frequency is increased to the 50.4688-Hz resonance frequency, the driving shaker is clearly in motion, and a pair of nodes may be seen (Figure 11) about an inch from the center. This distinction is characteristic of every pair of antiresonances and resonances. At every antiresonance, the driving DOF is stationary, and the shaker simply imparts a maximum force to the device under test (DUT). At each free-free resonance, the shaker moves freely and provides a minimum force to the DUT.

An Interesting Anomaly

A center-mounted beam is a seemingly ideal payload for a small shaker and an impedance head. Its symmetric mass distribution does not impose moments on the shaker's suspension or upon the dual-output sensor. However, it should be noted that small errors in symmetry (such as drilling the mounting screw-hole off center) will result in pairs of "split" resonances and antiresonances very closely spaced in frequency. This is because the center-clamped beam actually exhibits "repeated roots" for every mode. These are the "symmetric shapes" (where both beam tips move in phase) that a linear shaker can excite; and the "anti-symmetric shapes" where the beam tips move in phase opposition, requiring a rocking moment to excite.

The lack of perfect symmetry splits these repeated roots into pairs of distinct roots. In essence, we end up with two new families of modes, those associated with the "long cantilever" and a (very slightly) higher frequency set associated with the "short cantilever." Figure 12 illustrates measurement of such a flawed beam. This simple contrast demonstrates how readily base-shake antiresonance inspection can identify flawed assemblies and parts from a change in the detected frequencies.

Conclusions

The antiresonance is infrequently discussed in today's literature. It has much to offer the practicing dynamicist. The driving-point antiresonances of a *constrained structure* are the resonances that will result if the driving DOF is constrained not to move. This is a very useful predictive tool. Consider the obvious example of deciding where to position piping hangers or muffler supports. The antiresonances of a *free-free* structure in an ignorable-coordinate shake are the resonances of the same structure with the ignorable DOF grounded. This provides a simple means of identifying multiple natural frequencies without need to attach response sensors, a key to simple inspection for parts consistency and correct assembly. Further, the identification is not compromised by transducer mass loading; the only attachment is at a node for every (constrained) mode detected. **SV**