Quaternions – Correcting for Geometric Distortion in Dynamic Seismic and Satellite Testing

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Seismic testing sometimes requires significant roll, pitch, and yaw rotations from their MDOF vibration test systems when reproducing certain reference waveform vectors on seismic tables that are associated with particular seismic environments. When these reference waveforms (typically accelerations) are used for such testing, significant rotational motion of the test system's shake table can occur. As a result, the X, Y, and Z control transducers that are mounted on the surface of the table will no longer be aligned with the world (fixed) coordinate system as the table rotates about X, Y, and Z, but rather with respect to the shake table's body (moving) coordinate system. This lack of alignment with respect to the world coordinate system in turn causes geometric distortion in the output of the control accelerometers that are required to control such MDOF tests. These errors can cause further errors in the control system's ability to reproduce the motion specified by the test's reference waveforms with respect to the world coordinate system as required by the seismic test specifications. Similar problems can also occur in other such applications of MDOF testing with reference acceleration waveforms that similarly produces significant roll, pitch, or yaw rotations with their tests. This article presents methods based on the use of quaternions to correct the resulting control transducer measurement distortions. Quaternions are also used widely in physics, guidance and control, kinematics, robotics, autonomous cars, video games, and general graphics applications.

MIMO test waveforms sometimes require significant rotations in roll (about X – R_x), pitch (about Y – R_y), and yaw (about Z – R_z) from their MDOF test systems as a result of the spectral content of the particular reference waveforms used for the tests. (See References 1 and 4 for examples of 6-DOF test systems and applications.)

When high rotational displacement motion occurs, the control transducers mounted on the surface of the shake tables[1,4] that are used for this type of testing will no longer point in the world (nonmoving) coordinate system as the table rotates. Furthermore, the X, Y, Z values that are computed by time-invariant I/O transformations[1] typically used by MDOF control don't account for these rotations. Thus, the obtained estimates of the X, Y, and Z control responses[1,4] will exhibit the effects of significant geometric distortion.

Additionally, these time-invariant input transformations will not produce acceleration measurements that are aligned with the fixed world coordinates, but rather will produce measurements that are aligned with the moving-body coordinates of the table, which will rotate in 3D space as a function of the roll, pitch, and yaw rotational displacements that are associated with the test's particular reference waveforms. Reference waveforms are typically specified as acceleration waveforms obtained from the test article's field data.[2] Examples of such MDOF systems are shown in the illustrations that follow.

Figure 1 shows a 40-ton seismic table used for seismic simulation controlled by a Jaguar MIMO control system from Spectral Dynamics. Eight servo-hydraulic actuators arranged in a 2-2-4 configuration[4] drive the table with four vertical and two each (opposed) in the horizontal and lateral directions. Controlled operation is from 0.1 Hz to 100 Hz with maximum displacements of greater than 200 mm p-p. Loads of up to 60 tons can be accommodated. Typical applications are MIMO transient wave, MIMO swept sine, MIMO random, and MIMO replication for structural testing.

Some of the MIMO tests performed by this facility require significant rotational displacements. So the need for geometric distortion compensation has been a concern. However, the bulk of seismic test specifications have been 3 DOF, where MIMO control is used to suppress rotations[1,3,4] and thereby resolve the geometric distortion indirectly. As a result of the further use of measured field data,[2] the test requirements are starting to change where significant rotations are starting to appear in the newer test requirements. This has been another of the motivations to find better methods to address these newer requirements.

Figure 1. 6 meter × 6 meter seismic table with eight actuators – 4 Z, 2 X, 2 Y.

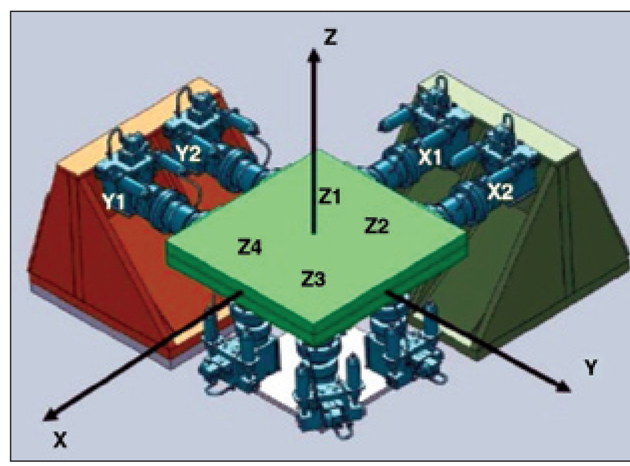

Figure 2. Eight-shaker arrangement with 2-X, 2-Y and 4–Z actuators.

Figure 2 shows a multi-shaker[3] system used for satellite testing, which includes a vibration table platform, which is 2.1 m × 2.1 m. The platform is excited by eight electro-hydraulic "hydrashakers" that are attached to the table by spherical couplings to allow for their independent motion. These are arranged in 2-2-4 configuration[4] that consist of two X, two Y, and four Z actuators. Each of the actuators is capable of providing 26,400 pounds of force dynamically. The static capacity of each actuator is 20 tons.

Figure 3 shows a typical dummy mass test specimen used in conjunction with MIMO swept-sine satellite testing, which simulates the satellite to be tested, including dimensions, total mass, center of gravity, and moment of inertia. It consists of two bolted sections, where the lower is a truncated hollow cone and the upper is 2,800 kg of solid steel. As a result, the dummy mass center of gravity is 3 meters from its base. It's attached to the vibration table platform via coupling ring. The four triaxial control accelerometers are mounted on the ring at 90° intervals around the ring. Test results obtained from X, Y, Z swept-sine testing of the dummy mass with

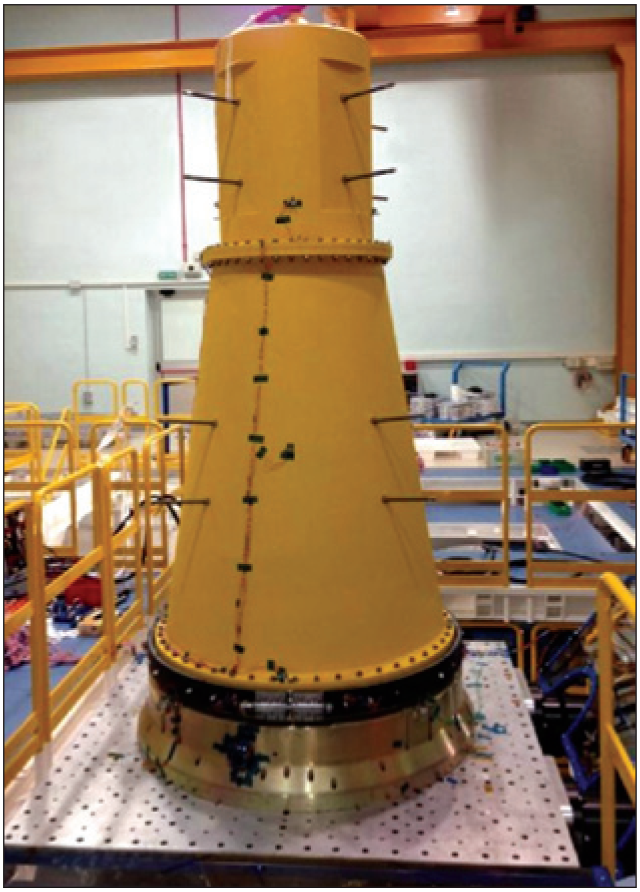

Figure 3. Typical four-ton dummy load used during system checkout.

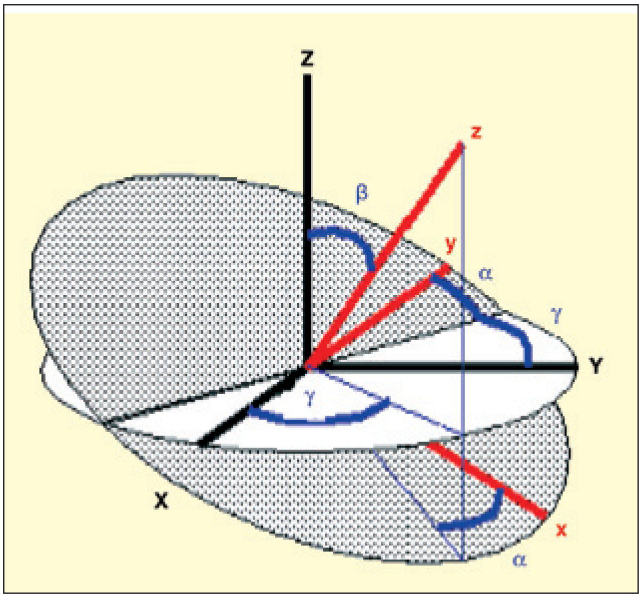

Figure 4. Rotation about X, Y, and Z.

the multi-shaker system and the Jaguar MIMO control system are discussed in Reference 3.

Reference 3 also details the amount of roll, pitch, and yaw that can occur while conducting single X and Y-axis swept-sine tests with the dummy mass, with MIMO and without MIMO control, where off-axis responses over 100% were observed typically without MIMO control as compared to tests with MIMO control, which were typically less than 10%. Large off-axis responses, such as those encountered without MIMO control, can cause significant roll, pitch, and yaw rotational responses with its concomitant geometric distortion.

Significant roll and pitch rotations can occur with a test article like the dummy mass, because large overturning moments can result due to its high center of gravity and cause significant pitch and roll moments during X and Y swept sine testing. MIMO control[3] can be used to suppress these rotations, which are typically not desired. However, there are other cases where it's desired to use such an MDOF system to excite pitch or roll modes during structural testing, or as is more common, the large roll and pitch rotations occur as a result of MIMO control not being used.[3] This is further motivation to better understand the effects of roll, pitch, and yaw rotations on the output of the control transducers and to develop methods to correct the geometric distortion that these rotations can cause.

In a more detailed fashion, Figure 4 illustrates the effects on the orientation of control transducers by showing the vibration tables shown in Figures 1 or 2 rotating about the X, Y, and Z axes. As can be seen in the figure, the Z transducers will no longer be pointing in the black Z-axis direction, but rather they will be pointing along the red Z-axis, which is pointing away from the true vertical direction as a result of rotations about the rotated X axis by α radians, and about the rotated Y-axis, by an angle of β radians. Additionally, Figure 4 also shows the table surface rotating about the black Z-axis by γ radians. These roll, pitch, and yaw rotational displacements of the shake table can cause geometric distortion in the thus rotated Z-axis transducer outputs as well as distortions in the associated outputs of the also rotated X and Y axes transducers if these rotations are large enough.

To get a sense of how large this error might be, we simulate moderate roll and pitch rotations, say around 10° each, and calculate what the components of a vertical acceleration would be that's in terms of body coordinates, which are tilted with respect to the world coordinates as a result of the induced roll and pitch displacements. We find that if we have pure Z motion of 1 g along the body coordinate positive Z axis and our table is tilted with 10° of roll and pitch, we would measure 0.97 g in the world coordinate Z axis from a Z accelerometer placed at the center of the tilted table and an even more troubling 0.1736 g and –0.1710 g contribution to each of the associated world coordinate X and Y axis. (That is, world coordinate aligned lateral accelerometers would see around 17% contribution from our purely vertical Z axis motion measured on the vibration platform). So significant errors start to appear for relatively small pitch and roll motions, which we've seen happen, for example, with satellite tests with high centers of gravity while undergoing slip-table X and Y swept-sine testing.[3]

Additionally, the shown rotations of α, β, and γ are functions of time. The use of time-invariant I/O transformations assumes that the X, Y, and Z transducers are aligned with the world coordinate directions at all times so are not a function of the time-varying roll, pitch and yaw motions that may occur. This approach is only reasonable if the control system is suppressing rotations,[1,3,4] which fortunately is the common case or if the resulting MDOF rotations are small and resulting distortion in the control transducer outputs is "low enough" to ignore. If this is not the case, a more general approach is needed. So a time-dependent I/O transformation capability is needed to deal with these more general types of MDOF motions.

The triplet of angles (α, β, γ), are called Euler angles.[5,6] They are also called roll, pitch, and yaw in the aeronautical and earthquake engineering fields. These are the names that we will use in the following discussion. These rotations are also useful to describe rotations or relative orientations of orthogonal coordinate systems. Unfortunately, their definition is not unique and in the literature,[5,6] there are as many different conventions as authors. The convention employed here is one of the more common ones. All rotations are in a counter-clockwise fashion (right-handed, mathematically positive sense).

Technical Approach

Euler Angles (Roll, Pitch, and Yaw). The Euler angles (α, β, γ) relate two orthogonal coordinate systems[5,6] having a common origin. The transition from one coordinate system to the other is achieved by a series of two-dimensional rotations.[5,6] In many cases, the rotations are performed about body coordinate system axes generated by the previous rotation step; the step-by-step procedure that is used is illustrated in Figure 4.

The convention used in the following discussion is that general-rotations will be represented by a series of rotations that are based on the aeronautical engineering convention. That is, a general rotation is represented by an initial yaw rotation, γ, about the Z axis of the initial coordinate system, followed by a pitch rotation, β, about the rotated Y' axis of this newly generated coordinate, and a final roll rotation, α, about the new rotated X' axis of the final coordinate axis. This convention is sometimes called rotation about body coordinates.

The result of this sequence of rotations is that the table's body coordinate system is rotated about some axis with respect to the world's coordinate system. However, the control accelerometers that are mounted on the table will now produce accelerations with respect to this rotated coordinate system. Thus, we need to transform accelerometer outputs, which are with respect to body coordinates, into their equivalent values with respect to stationary world coordinates. As a result, the main purpose of what we need to do is to transform the rotated body coordinate system of the vibration table back to its fixed world coordinate system counterpart to obtain corrected X, Y, and Z control accelerometer outputs that always point along world coordinate axes on a sample by sample basis for each of the acquired acceleration time histories.

To solve this problem, we will need to solve several associated problems. These are: 1) how to determine the representation of rotations in terms of Euler Angles; 2) to find an efficient computational form to represent the associated rotation; and 3) to determine how to use this representation to implement a coordinate transformation from the body coordinates of the moving table to the fixed world coordinate system. By solving these problems, we obtain an efficient transformation that will convert the measurements obtained from the table-mounted accelerometers into acceleration measurements along world coordinates.

Determining Rotations in terms of Euler Angles

The usual ranges for α, β, γ (roll, pitch, and yaw) are:

$0 <= \alpha <= 360^o$
$0 <= \beta <= 360^o$
$0 <= \gamma <= 360^o$

As discussed in References 5 and 6, rotations or transformations from one coordinate system into other coordinate systems are conveniently described by the triplet of Euler angles, (α, β, γ). With the use of the Euler angles, three-dimensional rotations can be dissected into a sequence of two-dimensional rotations, where in each rotation, one axis remains invariant.

Reference 6 shows that these 2D transformations can be extended to 3D, as shown in Equation 1 for a yaw rotation via:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \left[\mathbf{R}_z(\gamma)\right] \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} \qquad (1)$$

Equation 1 represents a yaw rotation of γ radians in terms of 3D matrices. 2D matrix formulations for pitch and roll can also be extended to obtain similar matrix expressions for pitch, β, and roll, α, in three dimensions. See references 5 and 6 for details on how this is performed and also on how to compose these basic rotations into the overall rotation resulting from sequential yaw, pitch, and roll rotations.

An example of the overall matrix that results when you perform successive yaw, pitch, and roll rotations is given by:

$$\left[\mathbf{R}_{xyz}(\alpha,\beta,\gamma)\right] = \left[\mathbf{R}_x(\alpha)\right]\left[\mathbf{R}_y(\beta)\right]\left[\mathbf{R}_z(\gamma)\right] = \qquad (2)$$

$$\begin{bmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta \\ \sin\alpha\sin\beta\cos\gamma+\cos\alpha\sin\gamma & -\sin\alpha\sin\beta\sin\gamma+\cos\alpha\cos\gamma & -\sin\alpha\cos\beta \\ -\cos\alpha\sin\beta\cos\gamma+\sin\alpha\sin\gamma & \cos\alpha\sin\beta\sin\gamma+\sin\alpha\cos\gamma & \cos\alpha\cos\beta \end{bmatrix}$$

The terms "yaw," "pitch," and "roll" (γ, β, α) are usually used in aeronautical engineering to describe a change in a plane's orientation, where yaw is a rotation about the vertical Z axis, pitch is a rotation about the left facing horizontal Y axis, and roll is a rotation about the front facing horizontal X axis. Figure 5 shows this convention.

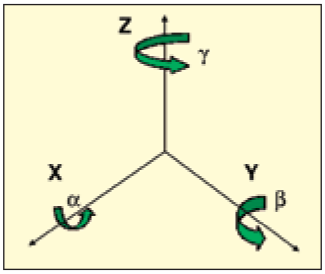

Figure 5. Roll (α), Pitch (β), and Yaw (γ).

However, if you want to use them to describe orientation absolutely, you have to decide on an orientation for which these values are 0. It's normally chosen as when the plane is level, not leaning, and facing to the east. To achieve an arbitrary 3D rotation, we first rotate the plane by the right values of yaw, then pitch, and then roll. Recall that the order is important because rotations don't commute.

For example, if you turn the plane first by pitch to 90° (so it's flying straight up), and then turn it through a yaw of 90°, it will be headed due north and leaning on its left side. Alternatively, if you turn the plane first in yaw by 90° (so it's flying due north), and then turn it through a pitch of 90°, it will be headed straight up, with the top of the aircraft toward due south. So, to calculate the absolute orientation of a body in space, a convention is needed on how to perform these changes in orientation in the right or standardized order. We will discuss the two most common conventions – moving axes and fixed axes.

So a general rotation in 3D can be described as a succession of 2D rotations, as discussed in the previous paragraphs, and which are first applied about the Z axis (yaw), and then about the thus rotated Y' axis (pitch), and finally about the resulting rotated X' axis (roll). This is called the moving-axes rotation convention, where the Y' axis is the new Y axis as a result of the yaw rotation, and X' axis is the new X axis that results from the previous yaw and pitch rotations, as shown in Figure 4. These are also called rotations about body axes.

There is another convention that is equivalent but where instead the rotations are successively applied about the fixed axes X, Y, and Z. This is called the fixed-axes rotation convention. They are also called rotations about the world axes.

Since in our Jaguar MIMO control system we measure roll, pitch, and yaw with respect to the body coordinates, we will use the moving-axes rotation convention to implement the method to correct the accelerometer measurements to compensate for the errors caused by the rotations of the shake table. As noted previously, we will use α to represent roll, β to represent pitch, and γ to represent yaw throughout this article.

General Rotation Matrices

There are numerous established methods[5,6] that use matrices to represent rotations. Although these are simple and straightforward, they have several intrinsic problems[6,7,8] that are due to certain singularities and numerical problems that are inherent in their formulation. For this reason, the method we will use to correct the accelerometer measurements will not use rotation matrices directly. Instead, the method chosen to implement the required rotations of the frames of reference will be based on results from quaternion algebra.[6,9]

Quaternions

The great Irish mathematician Sir William Rowan Hamilton discovered quaternions on October 16, 1843, as he was walking with his wife along the canals by the Royal Irish Academy in Dublin, Ireland. For many years, he had been searching for a way to multiply and divide "triples" of real numbers (today called 3D vectors) by extending the complex numbers, which allow the division of doubles (sets of two reals) into three dimensions. On that day, he realized that he needed three mutually perpendicular imaginary units forming a right-handed coordinate system, and one real instead of the real and two imaginary units that he had been using. Apparently, he was so excited by his discovery that he carved the fundamental quaternion algebra equations into a stone on the bridge with his knife. Today, a plaque containing his quaternion equations commemorates the spot where the original stone had been on the bridge.

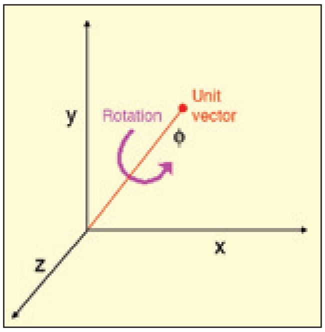

Figure 6. Rotation about a unit vector.

This section will introduce Hamilton's quaternions, their use to represent rotations, and how these can be used to compensate accelerometer measurements for the errors induced by rotation of the shake table. We'll discuss how quaternions can be used for this purpose, from an algebraic and geometric point of view, but with an emphasis on intuition so that the method can be better visualized and used effectively.

As Hamilton originally discovered, the quaternions are an extension of the complex numbers into four dimensions, with a real part and three distinct imaginary parts. Higher dimensional complex numbers such as quaternions are called *hypercomplex*. Many properties of quaternions can be discovered by extending the familiar theorems of complex-analysis with the use of a simple analogy to quaternions. As the complex numbers are used to implement rotations in 2-D, quaternions can be used to represent rotations in 3-D. Their main attraction is that they don't suffer from the formulation and numerical problems that the previous method based on rotation matrices have and as a result have found applications in many diverse fields.

Problems with Matrices Motivate using Quaternions

Distortion

- After several compositions (matrix multiplications), rotation matrices may no longer be orthogonal, which is necessary to be a rotation matrix, due to floating point inaccuracies.
- Matrix rotations also suffer from "gimbal lock," which can be caused by either cos(φ) or sin(φ) being zero for any value of roll, pitch, or yaw that in their matrix formulations, as in Eq. 3, that could arise during a test. This can also cause singularities in the matrix representation or a loss of rotational degree of freedom in the matrix representation of rotation.

Animation

It is not possible to implement a linear interpolation between two rotations (key frames):

- Matrices – given rotation matrices $\mathbf{R}_1$ and $\mathbf{R}_2$, $(1-t)\mathbf{R}_1 + t\mathbf{R}_2$ are not necessarily rotation matrices for all values of t.
- Unit vectors in axis angle representation (unit vector, angle of rotation) – given unit vectors (N_1,ϕ_1) and (N_2,ϕ_2), $(1-t)N_1 + tN_2$ are not necessarily unit vectors and $(1-t)\phi_1 + t\phi_2$ might give a zero or invalid angle.

To fix these problems, we need an approach for the representation of rotations such that it:

- Yields a method that is easy to normalize and that does not suffer excessively from numerical sensitivities or singularities in its representation of rotations.
- Provides for a method that allows us to perform linear interpolation of rotations in the correct space – the space of orthogonal linear transformations, which consists of only rotations.

Research[8,9] to find methods for dealing with these problems has resulted in the consensus that the use of quaternions to represent rotations provides such methods. This is the primary reason we've also chosen to use quaternions to implement the needed rotational transformations within the Jaguar MIMO control system from Spectral Dynamics to correct the accelerometer readings for the effects of the rotation of the shake table. The downside of this approach is that it is based on quaternion algebra and, as such, creates a need to learn at least the rudiments of quaternion operations for implementing the rotational adjustments that are needed. For these reasons, we'll provide a quick review in the following section.

Quaternion Algebra

Definitions

- $q = a + \mathbf{u} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is a quaternion, where a is scalar and $\mathbf{u}$ is a 3-D vector.
- A quaternion is an extension of the complex numbers to four dimensions, but with three imaginary units, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, which are also the familiar unit vectors along the right-handed X, Y, and Z axes that are found in the discussions of elementary vector analysis in 3D.
- The scalar a is called the real part, and the vector $\mathbf{u}$ is the imaginary part of the quaternion q. Vectors are represented by quaternions with a zero real part, the so-called "imaginary" quaternions.

Multiplication (Coordinate Free)

$$(a+\mathbf{u})(e+\mathbf{v}) = (ae - \mathbf{u}\cdot\mathbf{v}) + (e\mathbf{u} + a\mathbf{v} + \mathbf{u}\times\mathbf{v}) \tag{3}$$

which uses the ordinary "dot" and "cross" vector products of vector analysis: 16 multiplications; quaternion multiplication is associative, non-commutative, and distributes through addition and "almost" acts like complex number multiplication.

The quaternion conjugate is: $q^* = a - \mathbf{u} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$

Properties

$(q_1q_2)^* = q_2^*q_1^*$

$|q|^2 = qq^* = q^*q = a^2 + b^2 + c^2 + d^2$, which is the magnitude squared of a quaternion

$|q_1q_2| = |q_1||q_2|$

$$q^{-1} = \frac{q^*}{|q|^2}$$

if $|q| = 1$, i.e., a unit quaternion, then $q^{-1} = q^*$

The general case is as follows:

$$(a+\mathbf{u})^{-1} = \frac{q^*}{|a+\mathbf{u}|^2} = \frac{(a-\mathbf{u})}{|a+\mathbf{u}|^2}, \tag{4}$$

where $(a+\mathbf{u})(a-\mathbf{u}) = (a-\mathbf{u})(a+\mathbf{u}) = a^2+b^2+c^2+d^2$

Quaternion Multiplication of the Basis Vectors **i**, **j**, *and* **k**.

$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ {Equations Hamilton carved on the bridge stone}

$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$

$\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$

$\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$

{The last three are the same as the vector cross product}

Quaternion Multiplication of the vectors* u *and* v *as imaginary quaternions

Definition

- As discussed, vectors are defined as purely imaginary quaternions with a zero real part.
- Thus, the quaternion algebraic operations can be extended to vectors as:

$\mathbf{uv} = -\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\times\mathbf{v}$

$\mathbf{u}^2 = -\mathbf{u}\cdot\mathbf{u}$

Inverses

$$\mathbf{u}^{-1} = -\frac{\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}} = -\frac{\mathbf{u}}{|\mathbf{u}|^2}$$

Unit Vectors

$\mathbf{u}^2 = -1$

Unit Quaternions

- Unit quaternions have a magnitude of one.
- Unit quaternions are preserved under multiplication, i.e. $|q_1q_2| = |q_1||q_2| = 1$.
- Unit quaternions are also rotation quaternions.
- Rotation quaternions can be represented as:

$$\mathbf{q}_r = \cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\phi}{2}\right)\mathbf{N} \tag{5}$$

where **N** is a unit vector. Equation 5 is called the axis vector/angle representation of rotation quaternions and is illustrated by Figure 6, which shows how a rotation quaternion is used to rotate the coordinate system given by the X, Y, and Z axes about the unit vector **N** by ϕ radians.

Connection to Rotation

Axis Vector/Angle Rotation

N = Axis Vector

ϕ = Angle of Rotation

$$\mathbf{v}_{new} = \cos(\phi)\mathbf{v}_{old} + (1-\cos(\phi))(\mathbf{v}_{old}\cdot\mathbf{N})\mathbf{N} + \sin(\phi)\mathbf{N}\times\mathbf{v}_{old}$$

where $\mathbf{v}_{new}$ is the vector $\mathbf{v}_{old}$ after $\mathbf{v}_{old}$ has been rotated ϕ radians around the unit vector **N**. An example is shown in Figure 6, where the XYZ frame of reference that contains the vector $\mathbf{v}_{old}$ is rotated about the unit vector **N** by an angle ϕ.

Quaternion Rotation with the use of Sandwiching (Sandwich Product)

If **N** is a unit vector, then $q = \cos(\phi/2)+\sin(\phi/2)\mathbf{N}$ *is a unit or rotational quaternion.*

Then the vector **v** can be rotated into the vector **w** by the following triple quaternion product (sandwich product); length is preserved:

$$\mathbf{w} = \mathbf{R}_q(\mathbf{v}) = \mathbf{q\ v\ q}^* \tag{6}$$

Key Facts

$\mathbf{R}_q(\mathbf{v}) = \cos(\phi)\mathbf{v}+(1-\cos(\phi))(\mathbf{v}\cdot\mathbf{N})\mathbf{N}+\sin(\phi)\mathbf{N}\times\mathbf{v}$ is the rotation of **v** about **N** by ϕ radians.

$(\mathbf{R}_p\cdot\mathbf{R}_q)(\mathbf{v}) = \mathbf{R}_{pq}(\mathbf{v})$ (Composition of Rotations ⇔ Sandwich Product)

Comparison of Rotation Techniques – Composition

Vectors

$\boldsymbol{v}_{new} = \cos(\phi_1)\boldsymbol{v}_{old} + (1-\cos(\phi_1))(\boldsymbol{v}_{old}\cdot\boldsymbol{N}_1)\boldsymbol{N}_1 + \sin(\phi_1)\boldsymbol{N}_1\times\boldsymbol{v}_{old}$

$\boldsymbol{v}_{composite} = \cos(\phi_2)\boldsymbol{v}_{new} + (1-\cos(\phi_2))(\boldsymbol{v}_{new}\cdot\boldsymbol{N}_2)\boldsymbol{N}_2 + \sin(\phi_2)\boldsymbol{N}_2\times\boldsymbol{v}_{new}$

38 multiplications

Matrices

$R(\boldsymbol{N}_1,\phi_1)\,R(\boldsymbol{N}_2,\phi_2)$

27 multiplications

Quaternions

p q

16 multiplications

Trade-offs in the use of quaternions for implementing rotations

Advantages

- More compact representation: quaternion representation – four numbers; matrix representation, nine numbers.
- Faster composition: quaternion product – 16 multiplications; matrix product – 27 multiplications.
- Better for interpolating between frames (see *Slerp* in the following section).
- Avoids distortion – normalization function: $q \to \dfrac{q}{|q|}$ is simpler than matrix normalization.
- Numerically robust in the presence of floating-point, round-off errors.

Disadvantages

- Slower Rotation: $R_q(\boldsymbol{v}) = q\,\boldsymbol{v}\,q^*$ – 24 multiplications; $R(N,\phi)(\boldsymbol{v}) = R(N,\phi)\boldsymbol{v}$ – nine multiplications
- Difficult to compose with other transformations: non-uniform scaling, shears, projections cannot be represented as quaternions

Interpolating Unit Quaternions

Spherical Linear Interpolation (slerp)

$$slerp\ (q_1,q_2,t) = \frac{\sin((1-t)\theta)}{\sin(\theta)}q_1 + \frac{\sin(t\theta)}{\sin(\theta)}q_2$$

where $\cos(\theta) = q_1\cdot q_2$ (dot product between quaternions).

slerp maps unit quaternions to unit quaternions (along geodesics). This is a good way to interpolate between rotations, since each interpolation, as *t* varies, produces a rotation operator (unit quaternion).

Implemening Rotations Using Quaternions

As in Equation 2 for matrices, one can use the Euler angles to construct rotation quaternions, using the same conventions for the order of rotations about X, Y, and Z; where:

$$\begin{aligned} Q_x(\alpha) &= \cos(\alpha/2) + \sin(\alpha/2)\mathbf{i} \\ Q_y(\beta) &= \cos(\beta/2) + \sin(\beta/2)\mathbf{j} \\ Q_z(\gamma) &= \cos(\gamma/2) + \sin(\gamma/2)\mathbf{k} \end{aligned} \tag{7}$$

Equation 7 define the basic rotational unit quaternions, which represent rotations about X, by α radians; rotations about Y, by β radians; and about Z, by γ radians.

By performing the required quaternion multiplications shown in (Eq. 8), one can obtain the general rotation quaternion about body and world axes, as was done with rotation matrices in Equation 2.

Rotations about Body Axes

For the case of rotating about body axes, we have:

$$Q_{xyz}(\alpha,\beta,\gamma) = Q_x(\alpha)Q_y(\beta)Q_z(\gamma) = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k} \tag{8}$$

where:

$$\begin{aligned} q_1 &= \cos(\alpha/2)*\cos(\beta/2)*\cos(\gamma/2) - \sin(\alpha/2)*\sin(\beta/2)*\sin(\gamma/2) \\ q_2 &= \cos(\alpha/2)*\sin(\beta/2)*\sin(\gamma/2) + \sin(\alpha/2)*\cos(\beta/2)*\cos(\gamma/2) \\ q_3 &= \cos(\alpha/2)*\sin(\beta/2)*\cos(\gamma/2) - \sin(\alpha/2)*\cos(\beta/2)*\sin(\gamma/2) \\ q_4 &= \cos(\alpha/2)*\cos(\beta/2)*\sin(\gamma/2) + \sin(\alpha/2)*\sin(\beta/2)*\cos(\gamma/2) \end{aligned} \tag{9}$$

Which is the rotation quaternion that is obtained by a sequence of yaw, pitch, and roll rotations about the body axes, which compared with Eq. 2 is much simpler. Along with the sandwich operation shown by Eq. 6, this quaternion can be used to implement a transformation between body coordinates and world coordinates. As for matrices, the transformation given byEquations 8 and 9 will also map body coordinates to world coordinates. This is the quaternion that is used to transform the accelerometer measurements on the rotating shake table from body coordinates to world coordinates, thereby correcting the accelerometer measurements for the table's rotations.

Rotations about World Axes

Similarly, we also have for the case of rotating about world axes that:

$$Q_{zyx}(\gamma,\beta,\alpha) = Q_z(\gamma)Q_y(\beta)Q_x(\alpha) = p_1 + p_2\mathbf{i} + p_3\mathbf{j} + p_4\mathbf{k} \tag{10}$$

where:

$$\begin{aligned} p_1 &= \cos(\gamma/2)*\cos(\beta/2)*\cos(\alpha/2) + \sin(\gamma/2)*\sin(\beta/2)*\sin(\alpha/2) \\ p_2 &= \cos(\gamma/2)*\cos(\beta/2)*\sin(\alpha/2) - \sin(\gamma/2)*\sin(\beta/2)*\cos(\alpha/2) \\ p_3 &= \cos(\gamma/2)*\sin(\beta/2)*\cos(\alpha/2) + \sin(\gamma/2)*\cos(\beta/2)*\sin(\alpha/2) \\ p_4 &= -\cos(\gamma/2)*\sin(\beta/2)*\sin(\alpha/2) + \sin(\gamma/2)*\cos(\beta/2)*\cos(\alpha/2) \end{aligned} \tag{11}$$

Which is the rotation quaternion that is obtained by a sequence of roll, pitch, and yaw rotations about the world axes. As for matrices, the quaternion transformation described by Eqs. 10 and 11 can be used to map body coordinates to world coordinates.

Transformations Between Body and World Coordinates

With the above roll, pitch, and yaw rotation quaternions given by Eqs. 9 or 11 in hand, the problem of transforming a vector,{v}, originally in body coordinates, into a vector, {w}, in world coordinates is resolved. The additional variable needed to fully resolve this is to decide whether the roll, pitch, and yaw are about body or world axes.

Roll, Pitch, and Yaw about Body Axes

Using the previous quaternion discussions, a closed-form expression for the transformation of the vector $\{v\}=\{v_1,v_2,v_3\}^t$, which contains the raw acceleration measurements, into {w}, the corrected tri-axial acceleration measurement is obtained. The expression obtained from the use of Eqs. 6, 8, and 9 and the "sandwich" operation is given by Eq. 12, where the rotations are defined about body axes:

$$W=[0,w_1, w_2, w_3]= Q_{xyz}(\alpha,\beta,\gamma)[0,v_1,v_2,v_3]Q_{xyz}(\alpha,\beta,\gamma)^* \tag{12}$$

After some algebraic simplifications of the equations created by the evaluation of Eq. 12, in terms of q_1, q_2, q_3, and q_4; the rotated vector is found to be given by the vector portion of the quaternion W, $w=(w_1, w_2, w_3)^t$, we find that its components are given by Eq. 13 as:

$$\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} = \begin{bmatrix} (2(q_1q_1+q_2q_2)-1)v_1+2(q_2q_3-q_1q_4)v_2+2(q_2q_4+q_1q_3)v_3 \\ 2(q_2q_3+q_1q_4)v_1+(2(q_1q_1+q_3q_3)-1)v_2+2(q_3q_4-q_1q_2)v_3 \\ 2(q_2q_4-q_1q_3)v_1+2(q_3q_4+q_1q_2)v_2+(2(q_1q_1+q_4q_4)-1)v_3 \end{bmatrix} \tag{13}$$

Equation 13 can be used instead of Eq. 12. Eq. 13 is usually more convenient to use and it produces results equivalent to what Eq. 12 produces but in a much simpler fashion. This is because Eq. 13 uses fewer operations than what Eq. 12 requires when used directly. Thus in practice Eq. 13 is what is used. Additionally, since quaternion algebra is used to determine the coefficients of the matrix vector multiplication in Eq. 13, the numerical problems normally associated with using matrix algebra to obtain the overall rotation matrix are avoided.

Roll, Pitch, and Yaw about World Axes

As has been discussed, we can also obtain an expression for the transformation of the vector $\{v\}=\{v_1,v_2,v_3\}^t$ into $\{w\}$, where the rotations are now defined about world coordinate axes. It is obtained from Eqs. 6, 10, and 11, by a similar "sandwich" operation:

$$W=[0,w_1, w_2, w_3]= Q_{zyx}(\gamma,\beta,\alpha)[0,v_1,v_2,v_3]Q_{zyx}(\gamma,\beta,\alpha)^* \quad (14)$$

Where after some algebraic simplifications, as in the previous discussion, the rotated vector is found to be given by the vector portion of the quaternion W, $w = (w_1, w_2, w_3)^t$, and where its components are now given in terms of p_1, p_2, p_3, and p_4 by:

$$\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} = \begin{Bmatrix} \left(2(p_1p_1 + p_2p_2) - 1\right)v_1+2\left(p_2p_3 - p_1p_4\right)v_2+2\left(p_2p_4 + p_1p_3\right)v_3 \\ 2\left(p_2p_3 + p_1p_4\right)v_1+\left(2(p_1p_1 + p_3p_3) - 1\right)v_2 + 2\left(p_3p_4 - p_1p_2\right)v_3 \\ 2\left(p_2p_4 - p_1p_3\right)v_1+2\left(p_3p_4 + p_1p_2\right)v_2+\left(2(p_1p_1 + p_4p_4) - 1\right)v_3 \end{Bmatrix} \quad (15)$$

As before, Eq. 15 can be used instead of Eq. 14, as the use of Eq. 15 is likewise more convenient, and it produces results equivalent to what Eq. 14 produces; i.e., both formulations enjoy the same discussed advantages that quaternions provide.

Ensuring that $Q_{xyz}(\alpha,\beta,\gamma)$ and $Q_{zyx}(\gamma,\beta,\alpha)$ are respectively Rotation Quaternions

Note that the formulations of Eqs. 13 and 15 implicitly enforce the constraint: $|Q|=(q_1^2 + q_2^2 + q_3^2 + q_4^2)=1$, which ensures that either $Q_{xyz}(\alpha,\beta,\gamma)$ or $Q_{zyx}(\gamma,\beta,\alpha)$ are respectively rotation quaternions.[6] Because of this, Eq. 13 and Eq. 15 are the preferred manner by which vectors should be rotated, as this implicit enforcement of the unit quaternion constraint provides more numerical robustness. It is the manner by which the effects of rotations on the outputs of accelerometers during MDOF vibration tests are corrected by the Jaguar MIMO control system when there are large rotational responses required during the test.

Conclusions

1. Significant measurement problems with control accelerometers can occur when there are large (over 10°) roll, pitch, and yaw motions occurring on the vibration platform used to conduct vibration tests. In most cases, the control accelerometers are mounted on the vibration table and thus susceptible to the discussed geometric distortion errors.
2. If using a MIMO control and test system, MIMO can control the severity of undesired roll, pitch, and yaw motions to reduce these errors.
3. If the underlying vibration environment that is the subject of the test has large roll, pitch, and yaw motions, then geometric compensation, as discussed here, is needed to address and correct these errors.
4. The use of quaternion algebra and its ability to simply handle rotation is a powerful tool to construct the requisite transformations to rotate the body coordinate system to the world coordinate system, so that the control transducer outputs are obtained with respect to the stationary world coordinate system.
5. Rotation quaternions and their associated algebraic properties make their use self-normalizing and thus avoid the numerical and singularity problems that rotation matrices typically exhibit.
6. The use of quaternions enables an efficient, robust and, accurate geometric transformation algorithm that can be used for a real-time transformation capability that converts raw transducer outputs aligned with body coordinates into their corresponding outputs that are aligned with world coordinates as specified by many current MDOF vibration specifications and as described in this article.
7. The use of quaternions as described in this article is written about extensively in the robotics, graphics, manipulator design, mathematics, physics, and engineering literature. References 6, 7, 8, and 9 have discussions in these various areas and applications. In this article and in Reference 6, in a more detailed fashion, we show yet another application of these methods for MDOF vibration control and analysis.

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