Structural Dynamics Modeling – Tales of Sin and Redemption

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The great twentieth century mathematician, John von Neumann, once said, "At a great distance from its empirical source, or after much abstract inbreeding, a mathematical subject is in danger of degeneration. Whenever this stage is reached, the only remedy seems to me to be the rejuvenating return to the source: the reinjection of more or less empirical ideas." This wisdom is especially applicable to the field of structural dynamics. This article looks at the historical and empirical bases of key aspects of structural dynamic phenomena, including damping of materials and built-up assemblies, behavior of viscoelastic materials, interaction of structures and fluids, and general parametric uncertainties. Migration of misconceptions in engineering practice and, in particular, commercial software products are cited. Illustrative examples of the benefits of recollection of fundamentals in aerospace, marine, and civil applications are described.

Nomenclature

Symbol	Definition
[B]	Damping matrix
{F}	Force array
[K]	Stiffness matrix
[M]	Mass matrix
T	Kinetic energy
V	Strain energy
{q}	Modal displacement array
{u}	Physical displacement array
[Φ]	Modal matrix
η	Structural damping coefficient
ω_η	Natural frequency (rad sec)
ζ_η	Modal critical damping ratio

The broad discipline of modern structural dynamics modeling is the product of advances in strength of materials, theory of elasticity, and theory of structures,[1] and automated computational analysis (primarily the finite-element method).[2,3] Complementing the above "two legs of the stool" is a third leg, namely the wealth of empirical data[4] that are often forgotten or ignored. Here we will focus on several areas of structural dynamics modeling, primarily damping and structural joints, that suffer from widespread misconceptions, neglect, and errors (sin) that can best be remedied by recollection of their historical bases (redemption).

The popular notion of proportional damping is based on Lord Rayleigh's citation, which quoted directly from Theory of Sound[5] (Ch. 5, Par. 97) states, "The first case occurs frequently, *in books at any rate*, when motion of each part of the system is resisted by a retarding force, proportional both to the mass and velocity of the part. The same exceptional reduction is possible when *F (the dissipation force)* is a linear function of *T (kinetic energy)* and *V (strain energy)*."

In addition, Lord Kelvin during the late 1800s[1] introduced the notion of internal friction, which he concluded is *not proportional to velocity, as in fluids*. The Rayleigh proportional damping model, which generally does not follow experimental data, has been generalized by Caughey and O'Kelly[6] with a complicated, velocity dependent formulation. At the present time, Rayleigh proportional damping remains an option for time-domain structural dynamic analysis in most commercial finite-element codes.

During the 1960s, demands in the aerospace industry and memory limitations of digital computers led to the introduction of component mode synthesis (CMS) techniques,[7-11] which provided means for incorporating modal damping in structural dynamic analyses that bypassed the conceptual pitfalls of explicit (proportional) damping models. Component mode synthesis tacitly pointed to means for coupling of structural subassemblies at interfaces (joints), which have become routine in modern computer-aided engineering (CAE) software tools.[12-14] However, the ease of automation also provides ample opportunity for naïve errors, which ignore local flexibility at structural component interfaces.

The engineering demands introduced during the early years of aeronautics and the subsequent space age caused investigators[15-21] to engage in development of combined fluid-structure modeling techniques that relied heavily on empirical data. These activities led to a few isolated situations in which (fluid dynamic perturbation) damping forces were found to be velocity dependent and the overwhelming majority of situations pointing to damping forces that do not fit simplistic "velocity" dependent models.

Most notable are the contributions of Kimball and Lovell[22] and Becker and Foppl,[23] who independently confirmed Lord Kelvin's observation that structural damping forces are generally not proportional to velocity. In fact, the two teams concluded that damping forces appear to be proportional to displacement and in phase with velocity. This model was subsequently expressed in terms of complex variables by Kussner[24] and Kassner.[25] The appropriate, complex formulation of structural damping remains a standard for aeroelasticity[15,16] and vibroacoustics,[19,20] which are typically expressed in the frequency domain. Viscous (velocity-dependent) damping models persist in time-domain applications due to mathematical difficulties associated with complex variables outside the frequency domain.

A wealth of empirical data supports the notion of structural damping in metallic structures[26,27] that is proportional to displacement (strain) and in phase with velocity. Moreover, damping appears to be predominantly concentrated at joints. Important exceptions to the structural damping model are found in viscoelastic materials, shock and vibration isolators, and (welded, bolted, riveted, and bearing) structural joints. Alternative empirically based models have been developed to describe these phenomena in the time domain,[28-30] and recently Genta and Amati[31] introduced an approximate model for structural damping in the time domain that employs a general viscoelastic model.

Two key facts result from all empirical models of damping in solid structures:

- A simple, linear, velocity-dependent damping model does not appear to be physically appropriate.
- Localized structural flexibility is a close partner with joint damping.

Joint flexibility, as noted by many investigators, represents a strong influence on the modal characteristics of structural systems.[32] Physically consistent formulations of material damping and dynamic stiffness (especially for viscoelastic materials) and localized damping and flexibility at joints are essential for parametric sensitivity analyses, system identification, and structural damage assessments.

Structural Dynamics Modeling: Present State of the Art

The systematic development of structural mechanics theory,[1] finite element analysis,[2] and computer-aided engineering tools[3] has resulted in an engineering community characterized by high productivity and, in many cases, a blind faith in automation. The overwhelming majority of structural dynamic models describing behavior of systems are expressed in terms of the following matrix equations:

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$$[M]\{\ddot{u}\}+[B]\{\dot{u}\}+[K]\{u\}=\{F_e(t)\} \tag{1}$$

For many typical structures, the mass and stiffness matrices are appropriately assumed to be constant under "normal" operating conditions. Definition of mass and stiffness coefficients, for typical rod, beam, plate, and solid components, is quite well established,[1,2] especially when the system is composed of metallic materials and there are no significant interactions with other media (especially fluids). Formulation of the system-damping matrix, however, is generally not based on well-developed theoretical foundations.

Fortunately, there is a wealth of empirical data indicating that the dynamic behavior of many structures can be adequately described in terms of undamped modal vectors (in a relevant frequency band[4]), i.e.:

$$\{u\}=[\Phi]\{q\}, \text{ solutions of } [\mathrm{K}][\Phi]=[\mathrm{M}][\Phi][\lambda] \tag{2}$$

where

$$\left[\Phi^t\right][M][\Phi]=[\mathrm{I}],\ \left[\Phi^t\right][K][\Phi]=\mathrm{diag}\left[\omega_n^2\right],\ \left[\Phi^t\right][B][\Phi]=diag\left[2\zeta_n\omega_n\right] \tag{3}$$

Note that an explicit damping matrix that produces the (possibly empirical) damping coefficients may be constructed by the following operation,

$$[B]=[M\Phi][2\zeta_n\omega_n]\left[\Phi^t M\right] \tag{4}$$

This damping matrix is fully populated and does not have a theoretical basis as is the case for the mass and stiffness matrices. Moreover, the above triple product is not related to an explicit "viscous" theoretical damping matrix.

Proportional Damping Formulations

Most engineering organizations employ empirically based values for modal damping (ζ_n) and the modal approach for description of "viscous" damping, when practical, which circumvents difficulties associated with the lack of a theoretical damping matrix.

A variety of artificially constructed mathematical forms for the damping matrix have been defined over the past century. One form that has managed to find its way into most finite-element codes is known as proportional damping and is attributed to Rayleigh:[5]

$$[B]=\alpha[M]+\beta[K] \tag{5}$$

Application of the modal transformation on this matrix form results in the following distribution of modal damping, which does not resemble typical empirical data records:

$$\zeta_n=\frac{\alpha}{2\omega_n}+\frac{\beta\omega_n}{2} \tag{6}$$

Caughey and O'Kelly[6] introduced the extension of Rayleigh's proportional damping formula:

$$[B]=[M]\sum_{j=0}^{N-1}\alpha_j\left[M^{-1}K\right]^j \tag{7}$$

which results in the less restrictive distribution of modal damping,

$$\zeta_n=\left(\frac{\alpha_0+\alpha_1\omega_n^2+\alpha_2\omega_n^4+\ldots}{2\omega_n}\right) \tag{8}$$

While the Caughey and O'Kelly generalization of proportional damping (in the limit) permits any frequency-dependent distribution of modal damping, it has two distinct shortcomings:

- The damping matrix is fully populated, introducing computational inefficiencies for large-order dynamic systems.
- It does not permit radical differences in the damping of closely spaced modes that sometimes occur in actual physical systems.

At this point, Rayleigh's and Caughey-O'Kelly's proportional damping constructs clearly bring us to Von Neumann's point of a "great distance from its empirical source" and "much abstract inbreeding," resulting in this "mathematical subject . . . in danger of degeneration." Our only remedy for this situation is to return to empirical sources.

Assembly of Structural Dynamic Models

During the 1960s, structural dynamic models for aerospace systems taxed the capacity of digital computers and a series of component mode synthesis (CMS) techniques were developed to address the challenge. The variety of CMS techniques includes Hurty-Craig-Bampton,[7,8] MacNeal-Rubin,[9,10] and Benfield-Hruda.[11] These techniques define component structural dynamic models characterized by discrete physical boundary degrees of freedom that facilitate direct connection to adjoining system components. Moreover, CMS techniques provide an easy means for incorporating empirical modal damping (expressed as generalized viscous damping coefficients).

Computer resources and numerical analysis techniques have now surpassed the limitations of the 1960s, and the use of CMS techniques is no longer necessary. However, the integration of system design and analysis through computer-aided engineering (CAE) employs strategies for automated assembly of highly detailed component models to form system dynamic models.[12,13,14]

Both CMS and state-of-the-art CAE techniques provide a subtle yet serious opportunity for falling into Von Neumann's point of "degeneration." Specifically, the ease with which an engineer can automatically stitch components together often simultaneously eliminates opportunities for proper reconciliation of models with empirical data. In particular, the root cause of deviation of a structural dynamic model from reality is often found in component interface flexibilities that are "shorted out" by elementary stitching of component models.

Interaction of Structures with Fluid Media

The advent of the aircraft age in the early 20th century and the subsequent space age prompted the engineering community to address aeroelastic instability (flutter),[15,16] launch vehicle structure-propulsion system instability (Pogo),[17] and structure-control system instability.[18] In addition, vibroacoustic coupling phenomena in aircraft and spacecraft systems[19,20] represent a serious threat for fatigue failures.

During the same time period, safety issues associated with ocean installations (especially oil and gas production platforms) required an understanding of wave-induced structural loads.[21] All of these cited issues have required faithful, empirically based representations of:

- Structural dynamic damping.
- Models describing perturbed fluid dynamic phenomena (i.e., fluid mass, damping and stiffness effects).

Legitimate characterizations of velocity-dependent damping were defined on the basis of flow perturbations in ducts (especially for Pogo dynamic models[17]) and on immersed marine structures.[21] *In these exceptional situations, the damping matrix is roughly proportional to mass and velocity.*

While detailed discussion of fluid-structure interaction is beyond the scope of this article, it is important to note that progress in the above-cited applications was heavily influenced by close interaction of experimental and theoretical endeavors. This is attributed to the novelty of these challenges (in contrast with a perceived confidence in more "conventional" challenges in the field of "dry" structural dynamics).

Damping in Structural Assemblies

During the late 1920s, Kimball and Lovell[22] and Becker and Foppl[23] independently determined by experiment that damping in typical structures is simultaneously proportional to displacement (strain) and in phase with velocity. Shortly thereafter, Kussner[24] and Kassner[25] introduced the concept of complex structural damping, which appropriately describes the observations of Kimball, Lovell, Becker, and Foppl. In short, the mathematical description of damping in typical structures shifted from a theoretical (viscous) formulation:

$$[M]\{\ddot{u}\}+[B]\{\dot{u}\}+[K]\{u\}=\{F_e(t)\} \tag{9}$$

to a hysteretic formulation known today as structural damping:

$$[M]\{\ddot{u}\}+(1+i\eta)[K]\{u\}=\{F_e(t)\} \tag{10}$$

Note that structural damping and viscous modal damping coefficients are related to one another as:

$$\eta = 2\zeta_n \tag{11}$$

The contributions of Kimball, Lovell, Becker and Foppl represent a great contrast from the situation that resulted from Rayleigh's citation about proportional damping,[5] which quoted directly from *Theory of Sound* (Chapter V, Paragraph 97) states, "The first case occurs frequently, *in books at any rate*, when motion of each part of the system is resisted by a retarding force, proportional both to the mass and velocity of the part. The same exceptional reduction is possible when F (*the dissipation force*) is a linear function of T (*kinetic energy*) and V (*strain energy*)." It is ironic that Rayleigh appropriately pointed out the unverified status of proportional damping, yet much of the subsequent technical community succumbed to Von Neumann's state of "abstract inbreeding" and "degeneration."

Displacement Proportional Structural Damping

The thorough treatment of structural damping found in the text by Cremer, Heckl, and Ungar,[26] provides a wealth of empirical data along with a technical viewpoint that complements the prevailing, automated finite-element mindset. Three crucial features inherent in many structural systems are clearly noted in that text:

- "Solid" structures generally exhibit damping forces that are independent of frequency (displacement-dependent structural damping) for a wide range of building materials.
- Structural damping is often extremely low (η~10^{-4}) for individual, unattached structural members, such as bars, beams, plates, and shells. This is typical for steel, aluminum, and other "hard" metals; damping may be two orders of magnitude greater for lead, concrete, and brick.
- Structural damping in assemblies is often on the order of η~0.01, which is attributed to losses in (welded, bolted, riveted, and bearing-type) joints.

Typical values of modal damping for aerospace and other structures, compiled by Wada and Des Forges[27] and by this author, are summarized in Table 1 and Figure 1, respectively.

The probability distribution for all of these modal damping parameters (see Figure 2) indicates that more than 90% of damping values lie below $\zeta_n = 2\%$.

These data suggest that:

- The mean modal damping (ζ_n) of aerospace and similar structural systems is on the order of 1% with
- Lack of any systematic frequency trend over a four decade frequency band. Moreover,
- The Rayleigh proportional damping curve fit is clearly not representative of the collection of modal test data.
- Mode-to-mode variations in damping for each of the test articles are attributed to exercise of joints in each particular mode. (Note the test fixture mode for the Boeing ISS-P5, which is associated with localized deformation of two heavy steel plates.)

These observations are consistent with empirical damping data trends cited by Cremer, Heckl, and Ungar.[26]

Viscoelastic Material Behavior/Structural Joint Models

While the material model based on elasticity theory[1] and structural damping[22-25] describes the dynamics of many structural assemblies, alternative models (e.g., schematic "circuit" models illustrated in Figure 3) are required for viscoelastic materials, shock and vibration isolators, and (welded, bolted, riveted and bearing) structural joints.

The Kelvin-Voigt model[28] represents the most commonly assumed (and physically erroneous) behavior of structural materials (the correct behavior more closely follows Kassner's form,[25] Equation 10). The standard-linear solid model (SLS) introduced by Zener[2] describes the fundamental (observable) behavior of viscoelastic materials, as does the more general "Maxwell-Weichert" model.[28] Closely related to the Maxwell-Weichert model is the Iwan model, which describes fundamental behavior of structural joints exhibiting slip-friction characteristics.[30]

It is rather interesting to cite some historical notes related to the four schematic "circuit" models:

- According to Timoshenko's biographical sketch on Lord Kelvin,[1] "He observes that structural materials are not perfectly elastic and, in investigating this imperfection, he introduces the notion of internal friction, which he studies by examining the damped vibrations of elastic systems. From his experiments, he concludes that this friction is not proportional to velocity, as in fluids." This observation predates the contributions of Kimball and Lovell[22] and Becker and Foppl[23] by about 50 years. More importantly, it appears that the so-called Kelvin-Voigt model is named in honor of the two scientists (rather than being a direct product of their works).
- The SLS (Zener)[29] and Iwan[30] damping models are clearly attributed to the published works of the two respective authors.
- The Maxwell-Weichert model[28] is obviously named in honor of the two scientists, since this generalization of SLS (Zener) was introduced well after their lifetimes.

An illustration of viscoelastic behavior of a typical shock and vibration isolator used to protect electronic subassemblies is summarized in Figure 4.

The graphics on the left side of Figure 4 illustrate (a) magnitude

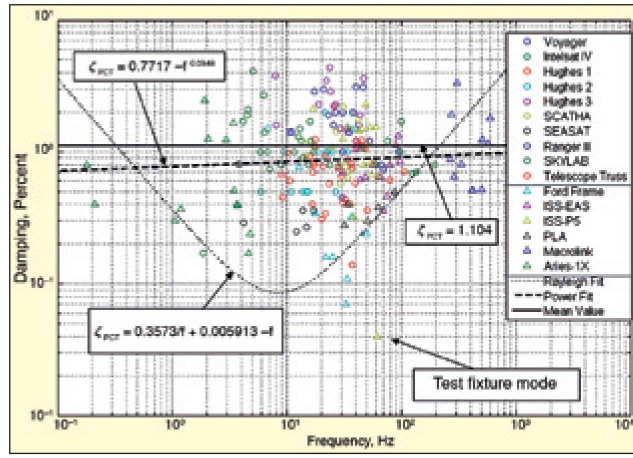

Figure 1. Typical values of modal damping.

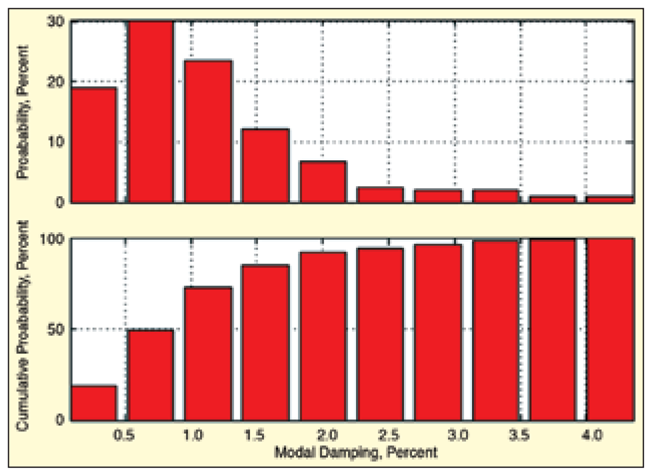

Figure 2. Modal damping statistics.

Table 1. Typical values of modal damping.

Source	Article	Frequency Band (Hz) Min	Max	Mean	Damping,ζ (%) Mean	STD
AGARD Conference No. 277 (Wada & DesForges) 1979 (Reference 27)	Voyager	10.60	52.90	28.41	2.08	0.74
	Intelsat IV (In Orbit)	1.85	195.00	82.19	0.86	0.39
	Hughes 1	9.01	47.91	27.62	0.83	0.40
	Hughes 2	5.89	40.15	22.27	1.03	0.49
	Hughes 3	7.95	46.88	29.20	2.28	1.01
	SCATHA	14.10	42.60	28.30	1.18	0.58
	SEASAT	3.77	16.99	10.93	0.39	0.21
	Ranger III	28.40	94.50	58.16	0.75	0.20
	SKYLAB	4.10	17.02	10.04	1.52	0.82
	Space Telescope Truss	16.60	106.00	53.50	0.57	0.16
Measurement Analysis Corporation (Coppolino) 2014	Ford Crown Vic Frame	12.72	49.79	30.75	0.36	0.34
	Boeing ISS-EAS	30.99	97.37	65.79	0.78	0.28
	Boeing ISS-P5	16.94	62.84	42.88	1.07	0.54
	Boeing PLA	32.04	88.19	53.73	0.37	0.09
	Macrolink Card Cage	269.28	596.96	407.65	1.21	0.86
	(Aerospace) Aries-1X FTV	0.18	4.78	2.55	0.79	0.69
Total of all Data Sets		0.18	596.96	52.98	1.10	0.78

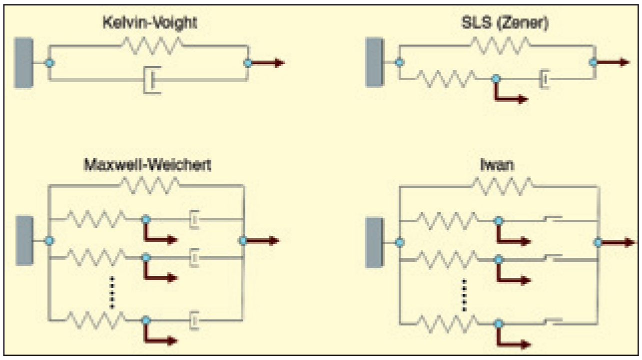

Figure 3. Schematic "circuit" models for viscoelastic and structural joint behavior.

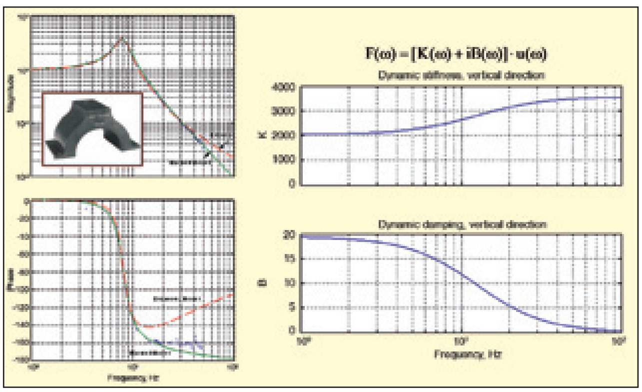

Figure 4. Measured behavior of typical viscoelastic shock and vibration isolator.

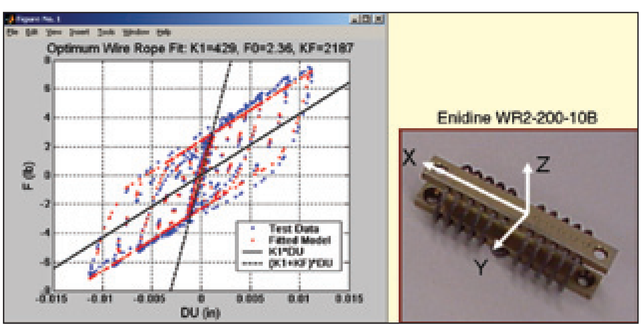

Figure 5. Measured (Z-axis) behavior of a typical wire-rope shock and vibration isolator. Low deflection stroke indicates high stiffness due to interwire "sticking," while larger deflection indicates low stiffness due to interwire sliding. Apparent stiffness and damping are clearly amplitude dependent.

and phase of frequency response estimated from broadband random data (blue curves), (b) curve fits for an erroneous Kelvin-Voigt model (red curves), and (c) curve fits for an appropriate SLS (Zener) model (green curves). The linear material model for the isolator, described in the frequency domain and illustrated on the right side of Figure 4, indicates that the effective stiffness and damping for the component are frequency dependent.

An illustration of slip-friction behavior of a typical wire-rope shock and vibration isolator (representative of many structural joints) is provided in Figure 5.

The wire-rope isolator is a revealing example of nonlinear characteristics of many structural joints. Not only is the friction damping nonlinear, but also the apparent stiffness may be very high for low-level loading and substantially lower for high-level loading. This type of behavior is an appropriate lead in to the subject of structural component interface constraints.

Fresh Look at Structural Damping

The primary aversion to employing structural damping in structural dynamic modeling is due to the inconvenience of complex matrix equations for time-domain analysis. This contrasts application of the complex formulation in aircraft flutter[16] and vibroacoustic[19,20] analyses, which are generally performed in

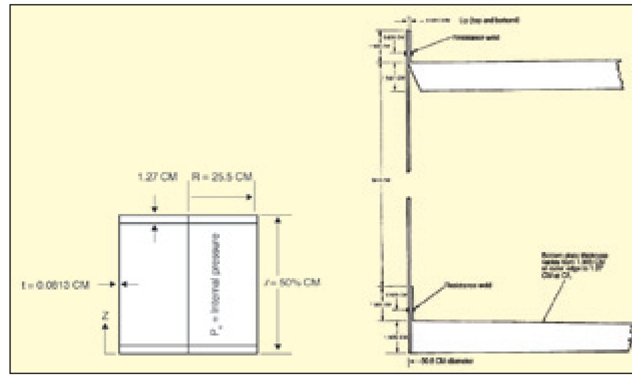

Figure 6. NASA Langley Research Center cylindrical shell test article.

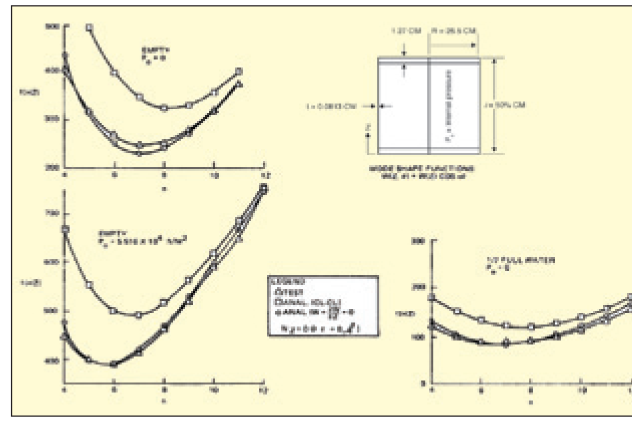

Figure 7. Comparison of predicted and measured shell breathing mode frequencies.

the frequency domain. Recently, Genta and Amati[31] published a state-space, time-domain formulation of structural damping, which exploits the Maxwell-Weichert model. The Genta-Amati formulation for structural damping provides a welcome approach to structural dynamic modeling in situations for which a modal formulation is not convenient. But it should be reiterated that the relationship between modal viscous and structural damping coefficients (Equation 11) is quite appropriate when the system can be described in terms of modal coordinates.

Interface Flexibility in Structural Assemblies

Mathematical solutions for structural components subjected to a wide variety of boundary conditions are a staple in the historical development of structural mechanics theory.[1] The mathematical solutions are complemented by a wealth of empirical data indicating variability of joint stiffness as well as damping (especially when joints have slip-friction behavior). Significant deviations from assumed ideal joint behavior are also present in structures that are composed of components welded to one another.

A quite revealing illustration of nonideal boundary conditions is noted in results of a series of modal tests conducted on thin cylindrical shells (see Figure 6) at NASA Langley Research Center in the mid 1970s.[32]

Initial NASTRAN mathematical models of the test article were defined with fixed end boundary conditions for all test conditions [(1) empty, unpressurized, (2) empty pressurized, and (3) half-filled with water, unpressurized]. Natural frequencies of shell breathing modes for the initial models were significantly higher than all corresponding test data. After changing the NASTRAN model end boundary conditions to pinned (shear diaphragm), which was not intuitively obvious to the young engineer (Coppolino), all of the analytical natural frequencies closely followed modal test data, as illustrated in Figure 7.

This lesson, experienced by many young engineers, is a clear example of non-ideal boundary conditions that exist in real structures. It is most unfortunate that this point is so easily missed by many practicing engineers due to the high degree of automation in day-to-day utilization of today's CAE tools.[3]

Conclusions

A mathematically convenient model for damping forces in structures, namely proportional damping, is attributed to Lord Rayleigh, who cited a hypothesis that "occurs frequently, in books." The proportional damping notion, moreover, assumes that damping forces are proportional to velocity, which Lord Kelvin during the late 1800s concluded is "not proportional to velocity, as in fluids" on the basis of experiments.

The engineering demands introduced during the early years of aeronautics and the subsequent space age caused investigators to engage in development of combined fluid-structure modeling techniques, which relied heavily upon empirical data. These activities led to a few isolated situations in which (fluid dynamic perturbation) damping forces were found to be velocity dependent and an overwhelming majority of situations pointing to damping forces that do not fit simplistic "velocity" dependent models. Most notable are the contributions of Kimball and Lovell, and Becker and Foppl, who independently confirmed Lord Kelvin's observation that structural damping forces are generally not proportional to velocity. This model was subsequently expressed in terms of complex variables by Kussner and Kassner. Unfortunately, mathematical difficulties with the complex model in time domain applications limited use of the appropriate "structural dynamic damping" model to aeroelastic stability and vibroacoustic applications.

A wealth of empirical data supports the notion of structural damping in metallic structures that is proportional to displacement (strain) and in phase with velocity. Moreover, damping appears to be predominantly concentrated at joints. Important exceptions to the structural damping model are found in viscoelastic materials, shock and vibration isolators, and (welded, bolted, riveted and bearing) structural joints. Two key facts result from all empirical models of damping in solid structures, namely (a) a simple, linear, velocity dependent damping model does not appear to be physically appropriate, and (b) localized structural flexibility is a close partner with joint damping. Joint flexibility, as noted by many investigators, represents a strong influence on the modal characteristics of structural systems.

Opportunities for erroneous structural dynamic modeling and analysis offered by simplistic models of damping and joint stiffness and automated CAE tools are remedied (or redeemed) by following John von Neumann's advice, namely, "At a great distance from its empirical source, or after much abstract inbreeding, a mathematical subject is in danger of degeneration. Whenever this stage is reached the only remedy seems to me to be the rejuvenating return to the source: the reinjection of more or less empirical ideas." A similar point of counsel, attributed to Augustine of Hippo (c 400 AD) suggests "that we should be willing to change our mind . . . as new information comes up." In the context of structural dynamic models, the apparently "new" information to some listeners is actually somewhat "old." Nevertheless, the empirical sources offer us a clear path from "sin" to "redemption."

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