How Heavy are Your Mode Shapes?

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Long ago and far away in an age when no analyst thought seriously about using a finite-element model to predict dynamics, in a place where test engineers couldn't spell FFT, we still performed modal analysis. Lumped-mass models were formulated to approximate the dynamic behavior of mechanical structures. Matrix iteration and other methods were used to identify the *normal modes* of these undamped models. Each normal mode was defined by *four* parameters:

- Natural frequency
- Viscous damping factor
- Nondimensional mode shape
- Modal mass

Ask any modern modal practitioner today for the modal results of a test or analysis and he will probably give you three parameters per mode. If you ask, "Where are the modal masses?" he is likely to look down his nose at you and say, "These are *orthonormalized* (unit modal mass or UMM) mode shapes – all of the modal masses are one." That smug and knowing expression is apt to fade from his face if you query, "But what are the physical dimensions or engineering units of those unit masses? One *what*?"

The sorry fact is that the *units* of modal mass are routinely overlooked by modern practitioners. The proper dimension of modal mass is not specified by any internationally recognized standard, and that compromises the value of any set of mode shape vectors for use in a team-executed project. If the analyst lives in a European or Asian *Système international d'unités* world (one where strict ISO metric is spoken), his structural FRF measurements were taken with a force channel calibrated to *Newton* (N) and an acceleration channel calibrated to *meter per second-squared* (m/s^2) units. As a result, the modal masses of his mode shapes (unit mass or otherwise scaled) are in units of kilogram (kg).

In contrast, an American colleague enters *mv/lb* and *mv/g* scale factors to his FFT analyzer. He measures force in pounds and acceleration in gravitational units or gs. His orthonormalized mode shapes are thus scaled to one lb/g (colloquially, one mass-pound). Now let's presume both of these competent practitioners test the same structure over the same bandwidth and that neither blunders in his test. They each produce virtually the same set of natural frequencies and damping factors and a set of orthonormalized mode shapes. The only difference in the two results is that every element in the American's vectors is larger than his compatriot's by a factor of $\sqrt{2.2}$.

In this age of international cooperative design, the difference between kilogram and mass-pound-scaled orthonormalized modes can mean the difference between project success and failure! Modal models of various components are commonly brought together from the test and analysis worlds to produce a dynamic model of the whole structure. Err the vector lengths for a single component and the total model will exhibit erroneous frequencies, damping and mode shapes. You can't balance your checkbook if most of the entries are in dollars but some are in an unspecified currency.

This little matter of modal mass units can bankrupt your expensive test and FEM collaboration. And that collaboration doesn't need to be international to suffer such failure. Any two dynamicists holding different opinions on (or not knowing) the mass *units* scaling of their mode shapes are sufficient to sour the cooperative result. Add in the occasional guy who likes his acceleration in inch/second^2 or his force in dynes, and the league of the unwashed who just don't bother attaching physical units to their FRFs, and the problem becomes obvious: having natural frequencies f_n damping factors ξ_n and mode shapes $\{\phi\}_n$ is not enough – you need the modal masses M_n *and their dimensional units* to have a complete model.

So what exactly is a modal mass? To answer that, I'll fall back to a simplistic undamped lump-mass model where N masses are connected by linear springs, some of which attach the "structure" to ground. Each mass is allowed to move in a single direction, the degree of freedom (DOF) x_j. The N DOF can be conveniently arranged in a vector $\{x\}$. The general equations of forced dynamic motion may then be written as $[M]\{\ddot{x}\}+[K]\{x\}=[F]$. Owing to Maxwell-Betti reciprocity and Beltrami-Michell compatibility, $[K]$ and $[M]$ are symmetric ($N \times N$) matrices. Further, $[K]$ is known to be *positive definite* if the modeled structure is statically stable (stands on its own). The N diagonal elements of $[M]$ are the physical masses, m_{jj} and the off-diagonal elements reflect any dynamic coupling between them. The $[K]$ matrix contains the stiffness terms reflecting the interconnecting springs, k_{jk} (and their static coupling) between the masses. For the special case of free sinusoidal vibration at frequency f these equations collapse to $[K]\{x\} = \omega^2[M]\{x\}$, where $\omega = 2\pi f$.

Now we can solve the free vibration problem using any number of numerical methods. One of the oldest, matrix iteration, actually predates the digital computer and adds some wonderful physical understanding of modal vectors. Pre-multiply the free vibration equation by the inverse of $[K]$ and divide it by ω^2, obtaining $1/\omega^2\{x\} = [K]^{-1}[M]\{x\}$. Note that the left side of this equation multiplies a mode shape vector by a scalar; the right side multiplies the same vector by a matrix.

Multiplying a vector by a scalar produces a vector with a different length but the same direction. Multiplying a vector by a matrix changes both the length and the direction. So, a solution vector of this equation is one that is *not rotated* by the matrix multiplication. A solution may be found by guessing a "normalized" vector $\{\phi\}$. The vector is normalized by dividing all its elements by the largest element value so that all contained DOFs are limited to the range ±1. This trial vector is multiplied by the *dynamic matrix* $[K]^{-1}[M]$, resulting in a new vector that is normalized in the same manner.

The largest vector element used for this second normalization is retained as a trial $1/\omega^2$ value. If the starting and final vectors are element-for-element identical (within acceptable error), the process stops. The vector is retained as the solution vector $\{\phi\}_n$ and its natural frequency is retained as $f_n = \omega_n/2\pi$. If the two vectors do not match, the computed vector is used as a new starting vector and the process is repeated. Eventually this iterative process will converge to the mode with the lowest natural frequency (largest $1/\omega^2$ *eigenvalue*). Once a solution vector (an *eigenvector*) is found, it is eliminated from further iterative consideration by use of its *generalized orthogonality* property. This permits further iteration to find the second and higher modes.

A complete solution to $[K]\{x\} = \omega^2[M]\{x\}$ consists of N *natural frequencies* f_n each matched to a mode shape vector $\{\phi\}_n$. If we stack these N solution vectors side by side, they form an $N \times N$ transformation matrix, $[\phi]$ that may be used to express any motion of the structure as a summation of motions in the N mode shapes. That is, $\{x\}=[\phi]\{q\}$ where $\{q\}$ is a vector of *modal participation factors*, the amount of each mode to add to the summation. If we substitute this definition in the equations of forced motion and then pre-multiply the equations by the transpose of the transformation, forming $[\phi]^T[M][\phi]\{\ddot{q}\}+[\phi]^T[K][\phi]\{q\}=[\phi]^T\{f\}$ something quite miraculous happens – both resulting matrices are diagonal; all of the off-diagonal elements are zero. Thus, the modal vectors uncouple the equations of motion, making each dynamic equation independent of all the others. The resulting "generalized" equations take the form $[M_{gen}]\{\ddot{q}\}+[K_{gen}]\{q\}=[F_{gen}]$. The N diagonal elements, M_n of $[M_{gen}]$ are the modal masses. To *orthonormalize* the mode shapes, we must divide each (±1 scaled) modal vector element by the square-root of its resulting M_n.

When orthonormalized modal vectors are used to generalize the equations of motion, the result becomes $[I]\{\ddot{q}\}+[\omega_n^2]\{q\}=[F_{gen}]$. That is, orthonormalized modal vectors transform the physical mass matrix to an identity matrix and the stiffness matrix to a diagonal matrix with ω_n^2 elements on the diagonal. So, each modal or generalized mass equals 1 mass unit, while the diagonal elements of the generalized stiffness matrix are the squared circular natural frequencies.

In contrast, performing the same type of *congruence transformation* using ±1-scaled modal vectors produces a generalized mass matrix with N different modal masses, each less than the total physical mass in the model. The resulting N different diagonal stiffness values in the associated $[K_{gen}]$ matrix have no obvious physical interpretation. At first glance, it would appear there is relatively little need to orthonormalize modal vectors. But this is not true – the experimentalist is actually damned to measure them!

Consider the analytic expression for the frequency response function (FRF) relating motion of DOF *a* to a force applied at DOF *b*. This may be derived from the generalized equations of motion by recognizing the relationships of Eq. 1. The result is Eq. 2.

However, when dealing with experimental measurements made from a real structure, no mass or stiffness matrices are known. All we know are the locations and directions of each DOF pair between which acceleration/force measurements are made. Further, the FRF measurement and subsequent curve-fitting does not separately identify the three frequency-independent terms ϕ_{an}, ϕ_{bn} and M_n – it merely returns the amplitude constant, A_{abn}. Given no other available basis for modal vector normalization, all of the M_n terms are assumed equal to 1, and the $\phi_{an}\,\phi_{bn}$ term is assumed to be the product of two orthonormalized modal coefficients.

Equations

$$\{\ddot{q}\}=-\omega^2\{q\},x_a=\{\phi_{an}\}^T\{q_n\}\ and\ \{F_{gen}\}=F_b\{\phi_{bn}\} \tag{1}$$

$$\frac{\ddot{x}_a}{F_b}=\sum_{n=1}^{N}\frac{\phi_{an}\phi_{bn}}{M_n}\frac{\omega^2}{\left(\omega^2-\omega_n^2\right)}=\sum_{n=1}^{N}A_{abn}\frac{\omega^2}{\left(\omega^2-\omega_n^2\right)} \tag{2}$$

$$\frac{\ddot{x}_a}{F_b}=\sum_{n=1}^{N}\frac{\phi_{an}\phi_{bn}}{M_n}\frac{\omega^2}{\omega^2-2\xi_n\omega_n\omega-\omega_n^2}=\sum_{n=1}^{N}A_{bm}\frac{\omega^2}{\omega^2-2\xi_n\omega_n\omega-\omega_n^2} \tag{3}$$

To determine the individual modal coefficients from a vector of $\phi_{an}\,\phi_{bn}$ products, a *driving-point* measurement such as $\ddot{x}_b\,/\,F_b$ must be measured and used as a reference. Taking the square root of its A_{bbn} value provides orthonormalized ϕ_{bn}, which may be subsequently divided into the remaining A_{abn} to determine the remaining orthonormalized elements of $\{\phi_n\}$.

For simplicity, I have chosen to present this explanation in terms of undamped real modes. However the results for damped systems, including those demonstrating complex modes, are entirely synonymous. For example, if the system is proportionately damped, that is if $[C]=\alpha[M]+\beta[K]$ (α and β being positive real constants), its eigenvectors, $\{\phi_n\}$, are identical to those of the undamped case, and forming the congruence transform $[\phi]^T[C][\phi]$ produces a diagonal generalized damping matrix $[C_{gen}]$ with diagonal elements of $2\xi_n\omega_n$ surrounded by zeros. Each ξ_n is a viscous damping factor (0 to 1) associated with the nth mode. The resulting acceleration/force FRF may be written as Eq. 3.

While the complex-mode solution of nonproportionately damped systems is mathematically more involved, it parallels the proportionately damped case discussed here. When such a system is experimentally tested, each $\ddot{x}_a\,/\,F_b$ measurement reflects amplitude (at each mode) proportional to the product of two complex-valued modal coefficients (complex vector elements) divided by a scalar constant with mass-proportional dimension.

If our purpose in testing a structure is simply to identify its resonance frequencies and damping and to admire a few animated mode shapes, we can live with an incomplete model. If the resulting modal model stays within the software system that generated it, we can still successfully play some "what if" games using *eigenvector modification* and simple *FEM-for-test* modelers. But if we want to contribute a component modal model to a larger modeling effort, our contribution must be complete and consistent with the larger model's metrics in all regards. In particular, the length scaling of any component model's mode shape vectors must be consistent with the host model to correctly convey the motion/force relationships of the added component.

Perhaps it is time to have an internationally accepted standard describing what constitutes a proper component modal model? Simply adhering to SDRC's 40-year-old specifications for a *universal file format* is a step in the right direction but falls short of the authority of a formal standard.

A few senior SEM members could get together at the next IMAC to start this process. People whose lives depend on combined FEM and modal test results might breathe a little easier knowing everyone involved in the dynamic characterization of new hardware fully understands how heavy their contributions are. SV

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