A Woeful Tale of Hidden Assumptions

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Inner monologue of a test engineer:

"All I've got to do is a modal test of a cantilever beam and correlate an FE model to the results? Can't get much easier than that. I'm going to pound this out, go home early, open a Sam Adams, and take it easy.

Alright, FEM is done, modes solved, now on to the test.

I've got my impact hammer; just need an accelerometer. I'll just use this triaxial accel and place it at the tip right here.

Now for some pretest checkout. Cables in place – check. DAQ ready – check. Arm warmed up for some serious impacting – ready. Let's do this!

. . . 1, 2, 3, 4, 5 . . .

No double hits; dang I'm good. At this rate, I'll have time for two beers.

Ok, IMAT, let's extract some modes.

And done. Ah, beautiful FRFs.

Wait, what's that?! Why is there motion along the axis with my bending modes? My FE model doesn't show any.

Ack, the accel must not be on straight. I guess I gotta do this again. One beer will have to do for the day.

Alright, accel reset, 1, 2, 3, 4, 5, processing . . .

What?! Again?! What's going on?!"

Enter the proverbial question: *Which is correct: the analysis or the test?* I'll give you a hint . . . in this case it's the test. So, what's wrong with the FE model? Well, nothing is wrong with the FE model, it is just loaded with mathematical assumptions that most engineers aren't aware of.

Let's take a closer look at the physics of the problem. Many of you will correctly guess that the axial motion seen in the test is due to beam shortening during bending (as well as tip rotation affecting the orientation of the accelerometer and a number of assumptions related to the test itself, but we'll ignore these for now). Take it a step further and you may also remember that the bending of finite element beams is typically formulated from Euler-Bernoulli beam theory, where bending deformation does not carry an axial component. We found the source of the error, right? Well, that's not the whole story; let's take a deeper look.

Instead of analyzing the whole beam using Euler-Bernoulli beam theory, let's chop it in half and apply beam theory to each half separately. The first half of the beam looks just like before: transverse displacement and rotation of the neutral axis with no axial displacement. Now the base of the second half is already rotated at the angle of the end of the first beam, so even though it won't exhibit an axial displacement itself, its tip will have moved in the axial direction of the undeformed beam, thereby introducing coupled bending and axial motion. Chop up the beam more and you'll get a better approximation of beam shortening. *Voila*, there's our axial deformation under bending.

So, we've proven to ourselves that the test results are reasonable; why then is the FE model wrong? You may be tempted to answer using my previous argument that the element formulation does not couple bending and axial motion. If you are tempted then you would be right – *if* you had only one beam element in your model, which I sincerely hope you don't. In all other cases, you would be wrong. Ask yourself this question: Wouldn't discretizing the beam FE model be the same as chopping up the beam and using Euler-Bernoulli beam theory? In short, the answer is no, at least for linear analysis. To understand why, we need to look further into the assumptions built into the standard finite element method (FEM).

It can be shown that beam shortening occurs primarily due to cumulative rotation of the beam, where the small $\sin\theta$ terms gradually add up. In the standard linear FEM, this effect disappears, because these terms are assumed to be small enough to ignore. On the other hand, if we use a geometric nonlinear FEM with large rotations, the effect reappears (as well as a whole lot of computational baggage). This, however, doesn't help us much when solving for modes. Why? First of all, modal analysis is a linear analysis, so we would need to linearize the nonlinear analysis about the current state, which we could do, but then we would just end up with the same linear formulation about the undeformed state as before and the same set of problems. Yes, we could perform a transient nonlinear analysis and process the response as we do in test – a valid effort – but then we would be spending more time doing the pretest analysis than we would doing the test. There is also such a thing as nonlinear modal analysis, but it is intended for different effects, and software tools generally don't include the option.

Another assumption that comes into play with the standard FEM is the strict locality of the shape functions. In other words, the shape functions associated with a given node are only associated with elements directly connected to that node. Furthermore, these shape functions are defined as displacements (or rotations) relative to the absolute coordinates of the undeformed configuration. Together, these formulation details mean that a series of beam elements in bending are not able to affect each other's displacements (i.e. displacements are not a function of displacements); only the force balance at common nodes is affected. In other words, a nodal degree of freedom at the tip has no idea what is happening anywhere else in the beam. The lack of beam shortening is therefore a consequence of the assumptions within the standard FEM formulation.

As an analysis tool, I've shown that the standard finite element method has its limitations, but all is not lost. You may have noticed that I've been using the word "standard" when referring to the finite element method. This is the mathematical tool we use all the time and the one that is implemented in all our favorite software packages, but it isn't the only game in town. Numerous formulations of the finite element method have been developed, some of which are helpful for this situation.

For example, I've recently been developing a new FEM variation I've termed the "relative" finite element method (RFEM), which formulates nodal degrees of freedom relative to each other rather than to absolute nodal coordinates. In this formulation, the degrees of freedom have no adherence or reference to a global or common coordinate system, which allows an element (as in the case of Euler-Bernoulli beams) to rotate along with the rotational degrees of freedom of the adjacent elements (i.e., it allows displacements to directly influence displacements.) The result is that we can realize beam shortening and other geometrically nonlinear effects in a more natural way than with the standard FEM.

The advantages of this new formulation don't come for free. It is inherently nonlinear, after all, which adds an extra degree of complication and requires a nonlinear solver. However, all nonlinear analyses perform a local linearization about the model's current state, which we can use to perform a linear modal analysis. This is the procedure I mentioned before that didn't fix our problem, but when it is applied to RFEM, the linearization retains the relative formulation, and therefore, effects such as beam shortening are seen in the modes. In the end, the lack of beam shortening in the standard FEM really isn't a nonlinear effect but a characteristic of the specific FEM formulation (the math model used) and the assumptions hidden within it.

Looking back on this woeful tale, we find several maxims that all engineers should be aware of:

- Just because you are getting a measurement you don't expect based on your pretest analysis doesn't mean that it isn't real.
- Your FEM results are only correct when delivered with the appropriate assumptions.
- A linear model can be wrong even in a linear regime.
- Most importantly, being aware of and understanding your assumptions can make the difference between a successful test and floundering with unanswered questions.

And what makes this tale so woeful? The test ran long and no beer was had.

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