Structural Stroboscopy – Measurement of Operational Deflection Shapes

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Operational Deflection Shapes (ODSs) can be measured directly by relatively simple means. They provide very useful information for understanding and evaluating the absolute dynamic behavior of a machine, a component or an entire structure. As understanding makes up most of the path to a possible solution, the visualization of the vibration behavior by Operational Deflection Shapes may guide the engineer to the point of the structure at which to make an optimal modification in order to control noise, control vibration, lessen fatigue, reduce wear or cure related problems. Modification decisions can be supported by one or a few frequency response measurements to check for the existence of resonance conditions.

Operational Deflection Shapes can also be predicted from a mathematical model (modal model), assumed boundary conditions, and operating forces if each of these is available. If, however, the objective is to study a particular structure under one or a few specific conditions, a direct measurement is faster, simpler, and more accurate than analytical prediction. No assumptions such as linearity have to be made.

The technique suggested in this article uses a two-channel measurement of complex transmissibility between a fixed reference transducer, and a second transducer moved sequentially to all points and directions of interest on the structure. At each frequency of interest the relative magnitudes and phases are extracted from the measurements. The magnitudes and phases when assembled in vectors (one for each frequency) represent the relative ODSs at each particular frequency. An absolute Operational Deflection Shape is obtained by multiplying the relative ODS by the absolute response, measured by the reference transducer.

A standard modal analysis software program can be used for data acquisition, extraction and animation of the ODS. It simplifies the data management, interpretation and evaluation of the results. The background and practical aspects of the measurement and analysis technique are discussed, and a number of practical problems are given as examples.

Definitions and Terminology. Let us take a close look at a simple structure that is forced to vibrate by an arbitrary excitation. Figure 3 shows a cantilever beam and $x(\xi,t)$ designates the continuous forced deflection function in one coordinate x as function of position ξ in space and time t.

If we want to describe the forced deflection, we turn to a finite discrete description, sampled in both space and time, rather than the continuous function. $\{x(t)\}$ designates the *Forced Deflection Shape (FDS)*, it is a vector where the elements represent the deflection time history for each defined point and direction or degree-of-freedom (DOF). Here x is an arbitrary symbol which may represent deflection in any coordinate. The ordinal position in the vector refers to the defined DOF on the structure.

Modal Decomposition. The FDS of a linear structure can be represented as a linear combination of its mode shapes.

$$\{x(t)\} = \sum_r \{\varphi\}_r q_r(t) \qquad (1)$$

The arbitrary FDS can be decomposed into its unique mode shapes by modal analysis. Or, if the modal parameters are already known it can be predicted by analytical simulation applying an assumed forcing vector to the modal model.

No! Mode Shapes do not Vibrate, Structures Vibrate!

Experience gained from discussions and from modal analysis tutorials show that the comprehension of modal properties vs. structural vibration response, measurements vs. physical reality is less than complete:

- ☐ What are the (absolute) magnitudes of the mode shape?
- ☐ Can you show an animation of the combination of several mode shapes?
- ☐ Will my structure resist this mode?
- ☐ How can you say my measurements are poor when they have such high coherence?
- ☐ How can there be a different interpretation of the coherence between vibration/force and vibration/vibration?

These and similar questions suggest that what engineers are basically interested in is how to determine the manifest structural vibration distribution; in other words, the *Operational Deflection Shape.* An Operational Deflection Shape designates the periodic motion pattern of a vibrating structure at a *specific frequency*, and under a *particular stationary operating condition.*

Applying a thought experiment: An Operational Deflection Shape represents the picture which is seen using a stroboscope, large, fast and powerful enough to freeze the object at a desired frequency, if the eyes of the observer are strong enough to resolve the (probably) very small deformations. Thus, an Operational Deflection Shape is an observation, or visualization, of particular dynamic behavior. It gives no indication of the inherent qualitative dynamic properties of the particular structure.

Note here that $\{\varphi\}_r$, the mode shape, is a time invariant description of the relative displacements (mode shapes do not vibrate), and is a qualitative inherent property of the system. The time variation is found in the scaling coefficient of Eq. (1), the generalized or modal response $q_r(t)$.

ODS Decomposition. The FDS may also be decomposed into the ODS:

$$\{x(t)\} = \{X\}_{f0} \sin(2\pi f_0 t) + \{r(t)\} \qquad (2)$$

$\{X\}_{f0}$ contains the sinusoidal part of the vibration, and a residual vector $\{r(t)\}$ contains the remaining vibration at other frequencies.

An ODS is not unique in the same sense that a mode shape is. It is dependent on the operating condition and on the choice of frequency, and is thus only valid for one particular condition. How well the ODS represents $\{x(t)\}$ depends on how purely sinusoidal the response is. For a very wide class of practical noise and vibration problems, experience shows that more than 95% of the vibration power is contained in a single frequency line, and hence the residual term $\{r(t)\}$ in Eq. (2) becomes negligible. For troubleshooting, this is equally true for periodic forced problems and for the case of randomly excited modes (resonance problem). If, however, the vibration spectrum contains more significant lines, e.g. harmonic components, Eq. (2) may be expanded to include a sum of ODSs.

Direct Measurement of $\{x(t)\}$. Measuring the set of continu-

Practical Vibration Measurements

Practical vibration analysis is used to study objects under particular stationary operating conditions. Our requirement is usually to establish a qualification reference or, in a trouble-shooting situation, to establish the cause of excessive noise or vibration.

In the problem identification phase, the first choice is often to measure a representative dynamic parameter, such as acceleration or sound pressure, and to make a spectrum analysis. Often we find that the spectrum, from a stationary process, exhibits one or a few predominant frequency lines or bands.

When the process forces are cyclical, which is typical for the free forces and moments from rotating machinery, the spectrum is usually characterized by one, or a few, predominant lines. Figure 1 shows a typical line spectrum from a structure excited by a rotating machine.

If the process forces have a random character, such as broad-band excitation from wind, sea waves or turbulent flow, most of the spectral power is likely to be found in one, or a few, narrow spectral peaks representing the modes of the structure. Figure 2 shows a spectrum from the same structure as Figure 1, but now with random excitation.

Whether a problem is caused by excessive operating forces, or is due to resonant amplification in the structure itself (to be verified from a frequency response measurement), the visualization of the dynamic deflection is a most desirable aid to the solution of the problem.

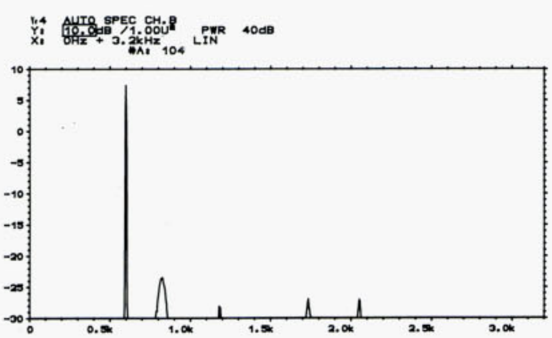


Figure 1. Output auto spectrum from a periodic process. Acceleration power spectrum measured at a structure excited by a rotating machine and dominated by the rotating frequency component.

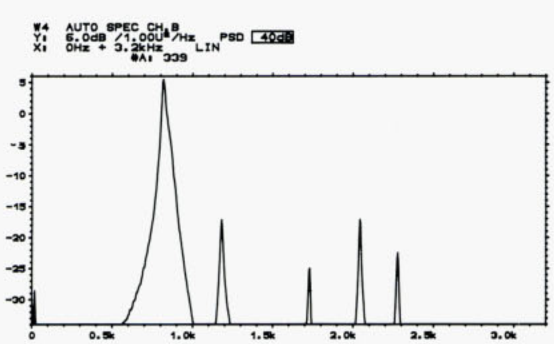


Figure 2. Output auto spectrum from a random process. The same structure as in Figure 1, but now excited by turbulent flow. Acceleration power spectral density dominated by the modal frequencies.

ous infinite time histories over the structure might be interesting for some non-stationary cases, e.g. transient response or a

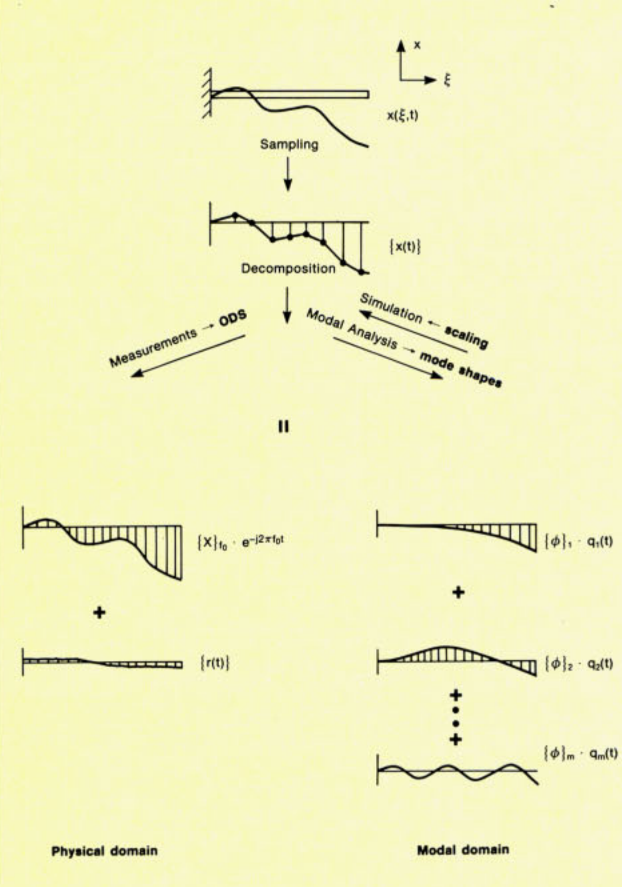


Figure 3. Decomposition of a forced deflection shape. The sampled FDS can be decomposed into its mode shapes or (for a periodic process) into ODS. The FDS may also be simulated from the modal parameters, assuming the forcing vector.

complex spectral composition, but would require large numbers of simultaneous response measurements and very high data processing capacity. Such a technique is outside the scope of this article but is discussed in Reference [6].

Mode Shapes vs. Operational Deflection Shapes. What is the difference? What is the similarity? From the previous discussion it is clear that an ODS is a linear combination of mode shapes. However, if the excitation frequency is close to a modal frequency and a structure's modes are well separated, the contribution from other modes may then become insignificant, and the ODS would have the same displacement distribution as the associated mode shape. The difference is that the ODS vibrates with absolute amplitudes, in contrast to the mode shape.

Unscaled mode shapes may be extracted from lightly coupled structures by the measurement of ODSs at the modal frequencies. The so called "mode studies" using speckle holography is one example of ODS measurements, and should not be mistaken for modal analysis. Randomly excited structures, e.g. offshore platforms, are another example where the mode shapes are estimated from ODSs. However, it must be kept in mind that such techniques will not provide a modal model since the scaling is unknown, and it only works for uncoupled modes as no decomposition technique can be applied. For modal analysis on ambient randomly excited structures see Reference [8].

How To Acquire Operational Deflection Shapes

The System Analysis Approach (Modal Analysis). Figure 4 illustrates two complementary approaches to acquire forced deflections or ODSs. The system analysis approach assumes system linearity and constitutes a hybrid technique. A full scale modal test yields the scaled mode shapes, the modal fre-

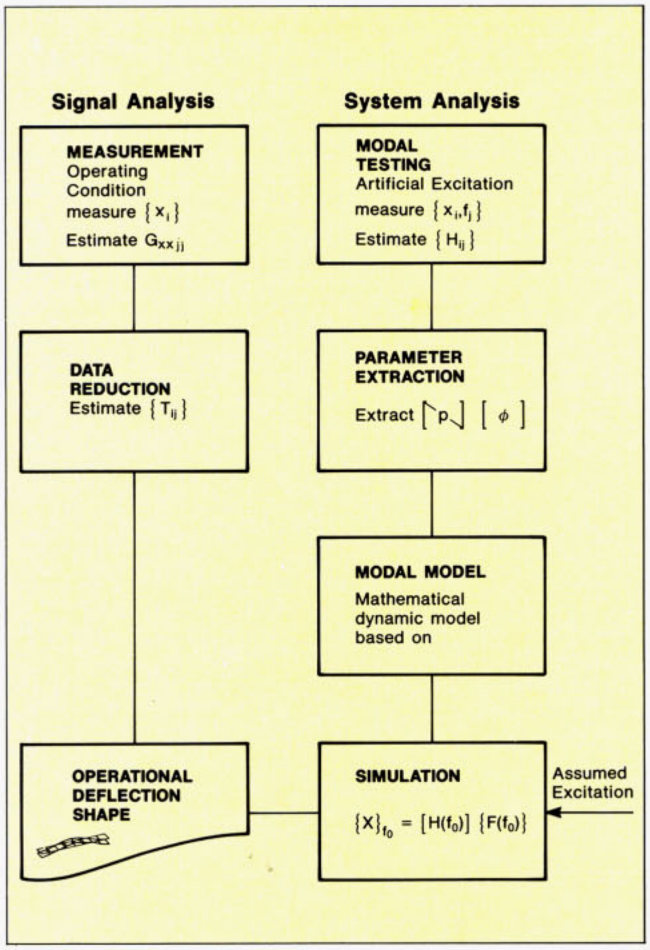


Figure 4. The two approaches to the Operating Deflection shape: 1) signal analysis by direct measurements of response and estimating the transmissibility functions during specific operation; 2) simulation assuming operating forces, using a previously derived modal model.

quencies and damping. These are the parameters for a complete dynamic mathematical model, i.e. the modal model. In other words we are determining and modeling the structure's inherent dynamic properties. The FDS or ODS can be analytically simulated, by loading the model with assumed operating forces.

A major difficulty in the application of this technique is to establish representative forcing functions, i.e. the quantitative excitation forces defined in space, magnitude and phase. Difficulties notwithstanding, this technique produces a dynamic model which can be used for simulations of multiple conditions. It provides a very useful analytical tool for many applications, including transient response and simulation of physical structural modifications.[1]

The Signal Analysis Approach (Transmissibility Measurements). The direct measurement of ODSs in a stationary condition is more appropriate in a number of situations: 1. when only one or a few operating conditions are of interest, the modal test can hardly be justified; 2. when the operating forces are nonobservable, or are analytically indeterminable, we will not know how to load the model; and 3. when the structure is suspected of being significantly nonlinear, and a parametric modeling technique will generally not be available.

The direct determination of ODSs requires the measurement of a set of response signals, spatially distributed over that part of the structure which is of interest. The responses will generally be measured sequentially, unless a very large measuring system is available and simultaneous measurements are required.

Using a two-channel analyzer, a common reference can be incorporated in order to obtain: 1. relative amplitude and phase information; and 2. compensation for small variations in response level during the measurement sequence.

In a practical measurement, the signal from a fixed motion transducer is used as a reference, and a second (roving) motion transducer is moved sequentially to the other test defined DOFs. The observed parameter is the *(Complex) Transmissibility (Function)* representing the relative deflections over the structure. This technique is simple and produces a result for one particular condition. The result is quantitatively and qualitatively only applicable to this particular measured condition.

Estimating the Complex Transmissibility Function. We shall define the complex transmissibility as:

$$T_{ij}(f) = \frac{X_i(f)}{X_j(f)} \qquad (3)$$

where j is the reference DOF, and i is an arbitrary DOF. Hence $T_{ij}(f)$ is the ratio between the Fourier spectra of a test DOF and the reference DOF.

Inspecting Eq. (3) shows:

$$T_{ij}(f) = \frac{\sum_k H_{ik}(f)\, F_k(f)}{\sum_k H_{jk}(f)\, F_k(f)} \qquad (4)$$

For the single input situation, $T_{ij}(f)$ is the ratio between two FRFs. It describes the dynamic properties of the path between the DOFs i and j, and the operating force (which cancels out). For multiple inputs, $T_{ij}(f)$ may be considered as the ratio of two linear combinations of the operating forces spectra.

Inspection also shows that $T_{ij}(f)$ is not affected by variations in the forcing level assuming a linear structure and assuming that the force *distribution and phasing* are constant. This is an important property in practical measurements as averaging can remove small fluctuations in the loading condition.

Practical estimation of $T_{ij}(f)$ can be made by using any of the FRF estimators. As the coherence is expected to be unity (see later discussion) any estimator must produce the same result at the frequencies of interest. For example, using the H_1, $T_{ij}(f)$ is calculated as the ratio of the *averaged* output cross spectrum between i and j and the reference auto spectrum in j:

$$T_{ij}(f) = \frac{GXX_{ij}(f)}{GXX_{jj}(f)} \qquad [= H_1(f)] \qquad (5)$$

Interpretation of the Transmissability Function. What does the transmissibility tell me about the system? In isolation, very little, and don't believe that a peak in the function means a structural resonance! We must keep in mind that the transmissibility represents the ratio between two spectra or two differently weighted sums of the operating forces (the weighting functions being the individual FRFs). Hence, a peak in the transmissibility only means a notch in the denominator spectrum (or an antiresonance in the denominator frequency response). A notch in the transmissibility just means a notch in the numerator spectrum. Figure 5 illustrates the principles, using examples, measured on a simple, but real structure. **Corollary!** The transmissibility gives the relative deflection between two DOFs as a function of frequency. *It depends on the forcing distribution* and is therefore not a system property!

What Can The Coherence Be Used For? Not very much! The coherence produced by a Fourier analyzer in association with a transmissibility measurement is:

$$\gamma^2_{ij}(f) = \frac{|GXX_{ij}(f)|^2}{GXX_{ii}(f)\, GXX_{jj}(f)} \qquad (6)$$

an expression of the linear relationship (causality) between the *measured signals* in the two channels.

It can be argued that the correlation between the two signals must be 100% at all frequencies since they are caused by the same action (force). If the coherence is less than unity at certain frequencies (and in practice it always is), it is due to measurement condition, e.g. too little signal at some frequencies (notches or very low level in one of the spectra), allowing the electronic and computational noise to become dominant.

The coherence must be unity at the frequencies of interest

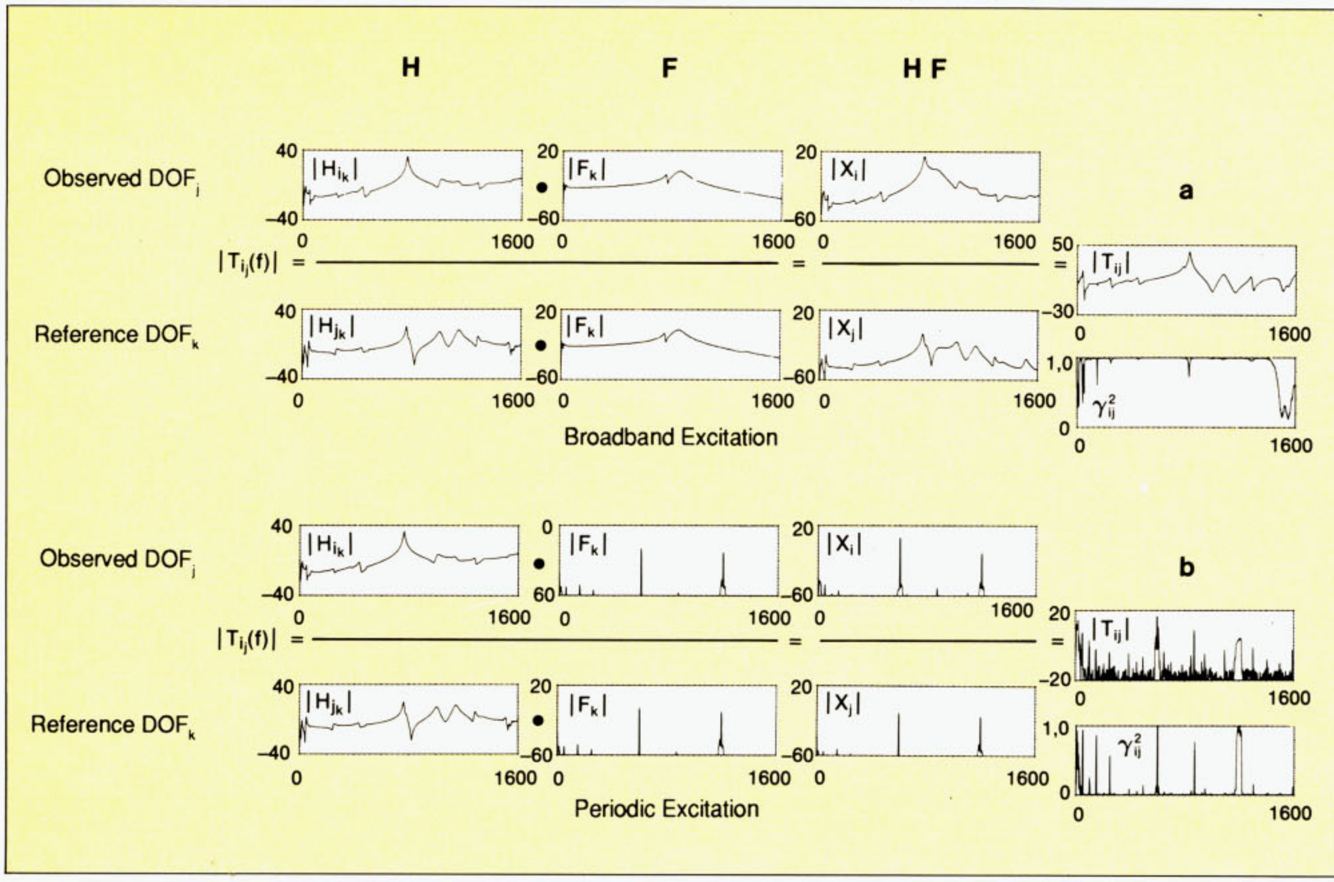


Figure 5. Graphical interpretation of the transmissibility function. The transmissibility is defined as the ratio between two output (Fourier) spectra which in turn are sums of the input force spectra weighted by the associated frequency response functions. The graphs show the features of the spectra from a single input, measured with: a) wide band excitation; and b) harmonic excitation. b) indicates that the transmissibility is defined at only two discrete frequencies where signal power is present.

unless a very gross error is introduced in the measurement. (E.g.: one of the transducers fell off, or the cable broke; the operating condition is changed during the averaging, i.e. if the excitation is acting in more than one DOF and the force distribution is changing, then the weight of the operating forces will change and hence the linear relationship will change.)

Corollary! The coherence gives information only about the actual measurements. It tells nothing about the system properties, e.g. nonlinear effects, *but does indicate if the excitation distribution has changed during the averaging.*

How to Measure ODSs Directly

Geometry. First the points and directions (the DOFs) of interest are defined on the structure. The number of DOFs and their spatial distribution (spatial resolution) can be chosen freely to fit the application. This is in contrast to a modal test where DOFs must be defined with sufficient density and representative distribution to ensure orthogonality between extracted modes.

In some cases only two DOFs are sufficient to determine, for example, the stress in a connection. Other applications may require any number of DOFs, depending on the complexity of the ODS and the desired spatial resolution.

Since the reference DOF is factored into all the other DOFs, careful consideration must be given to the choice of the reference. A natural choice would be a point of maximum response; this ensures the best possible signal to noise ratio for the measurements.

Operating Condition. Before the measurements are made, the specific operating condition must be established and precautions taken to keep it constant during data acquisition.

Measurement Parameter. The ODS will be the same whichever vibration parameter is measured, displacement, velocity or acceleration. The natural choice is to measure acceleration since it is less dominated by low frequency rigid body motion. Furthermore, accelerometers are the best transducers in terms of signal to noise ratio, dynamic range, directivity, etc.

Transmissibility Measurement. With a two-channel FFT analyzer we can use the standard FRF measurement technique for the transmissibility. The signals from the roving and the reference transducers have a strong linear relationship, as they are both outputs from the common excitation mechanism. This linearity is not dependent on the linearity of the structure itself, providing that a stationary condition exists. Hence, any FRF estimator based on averaged spectra can be used with identical results. However, as no curve fitting or noise reduction technique can be applied to the data later on (as no model for the data is available), all precautions must be taken in the data acquisition phase.

Calibration. Before starting to make the measurements, a calibration ratio check is recommended. The procedure is to move the roving transducer to the reference DOF (i.e. making $i = j$), and measure $T_{ij}(f_0)$. Correspondence between the measured and the expected function, with unit modulus and zero argument, confirms that the transducers, conditioners and the estimator are all matching.

Computer Aided Measurements. During the measurement, ODS extraction and documentation, a standard modal analysis software package is very helpful. A typical test procedure would be:

1. Data Acquisition – measurement and storage of the defined set of transmissibility functions.
2. Data Reduction – as no mathematical model for the transmissibility exists, no curve fitting of the data is possible. The software can help to extract the complex ratios at the desired frequencies using a "peak picking" procedure,[5] which is simply a readout of magnitude and phase at the desired frequency.

Documentation. The extracted set of transmissibilities constitutes the relative ODS:

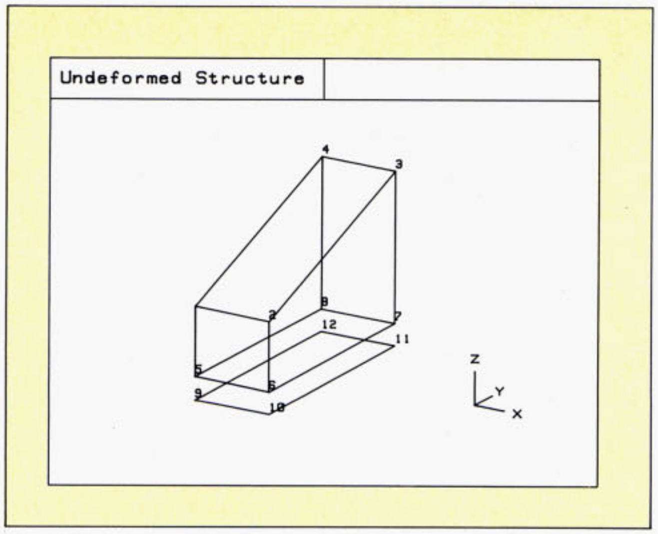


Figure 6. Geometrical model of a compressor unit.

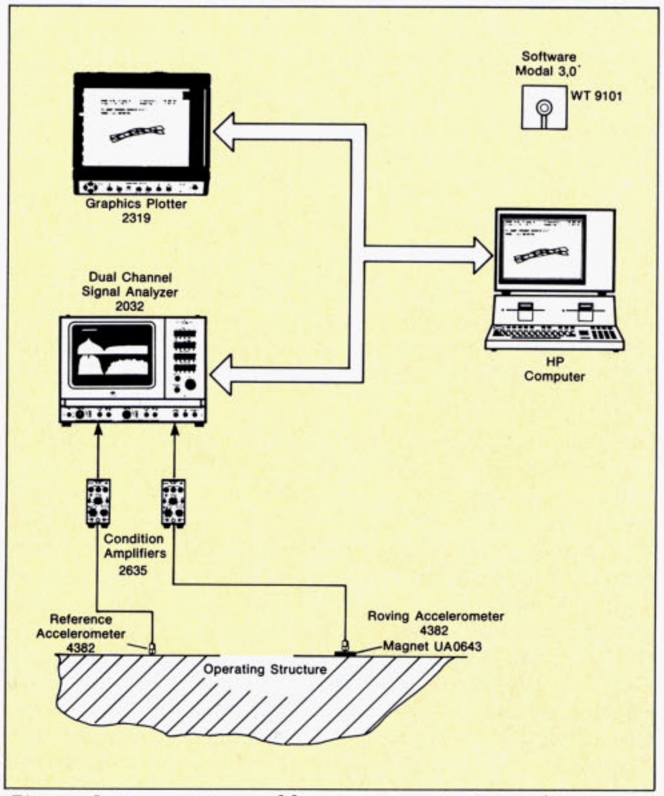


Figure 7. Instrumentation used for measuring, extracting and documenting ODSs.

$$\{X\}_{f0} = \{T_{ij}(f)\} \Big|_{f=f0} \tag{7}$$

The software provides graphical animation and plotting. The animation is a very attractive tool to visualize complex dynamic behavior. A list of the absolute operational deflections can be produced by multiplying each ODS by the corresponding RMS value[4] of the associated line (f_0) in the reference auto spectrum $G_{jj}(f_0)$:

$$\{x(t)\}_{f0} = \{X\}_{f0} \cdot \sqrt{GXX_{jj}(f_0)} \tag{8}$$

If parameters other than acceleration are required, arithmetic operation on the ODS can be used:

$$M_{ij}(f) = A_{ij}(f)/(j2\pi f) \qquad H_{ij}(f) = A_{ij}(f)/(j2\pi f)^2 \tag{9}$$

converts accelerence to mobility and compliance respectively.

Measurement of the ODS of a Rotating Machine

Objective. During operation of a rotating machine (5 kW compressor), repetitive fatigue in some piping connections was experienced. A spectrum analysis of the vibration signal showed one predominant peak at 56.5 Hz, equivalent to the rotational frequency. As the general vibration level was acceptable, it was decided to measure the associated ODS, first to identify the problem, and second to find a better location for the connections.

Geometry and Measurements. The frequency (56.5 Hz) was sufficiently low to assume that the machine structure would remain rigid. Hence only motion from the 6 rigid body modes in relation to the mounting base were to be considered. If rigidity is assumed, only 6 DOFs are theoretically needed to de-

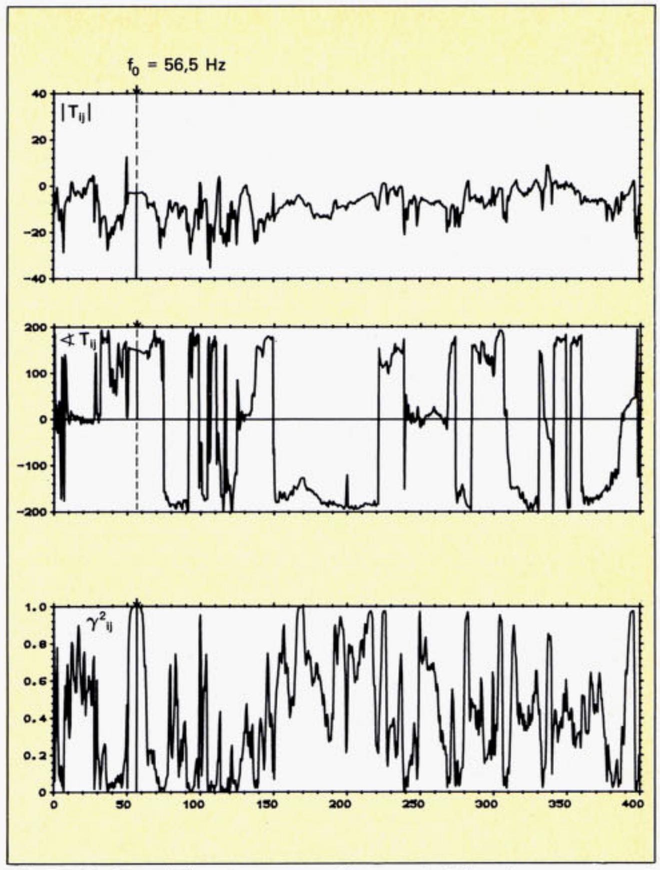


Figure 8. A typical measurement of transmissibility (magnitude and phase) and the associated coherence from the tested rotating machine. The cursor points to the rotational frequency and the function shows the typical plateau around it.

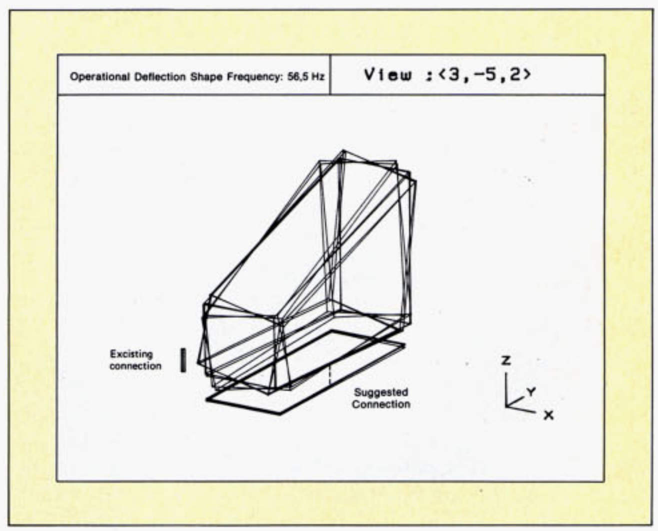


Figure 9. ODS at the rotational frequency of 56.5 Hz for the compressor example. The solution to the problem was moving the external connections to the position of minimum deflection.

scribe the motion of the machine. Redundant DOFs, however, can be used to improve the results by statistical inference. The DOFs defined were: 3 orthogonal at each of the 8 main corners of the machine, 4 vertical at the base of the resilient mounts and 3 orthogonal at the existing piping connections, making a total of 31 DOFs. The geometrical model, with the measurement points indicated, is shown in Figure 6. The instrumentation used is shown in Figure 7.

Figure 8 is a typical transmissibility function $T_{ij}(f)$, with the associated coherence function. The functions shown are typically noisy away from the dominant frequencies where they have defined plateaus. The ODS at 56.5 Hz. is shown in Figure 9. The plot represents the structure with the ODS superimposed at different time intervals, to give an impression of one full vibration cycle. It appears that the deformation shape truly represents a combination of rigid body modes, and that the resulting motion is complex in a modal sense (no nodelines).

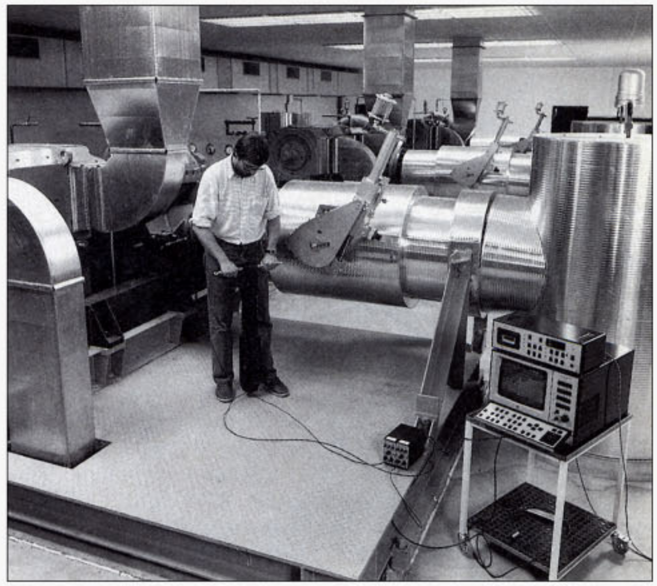
The relative displacement between the machine and its mounting base is clearly seen in the plot, and reveals the cause of the fatigue problem. A nearly maximum relative deflection over the piping connection is observed, causing excessive stresses. From the same plot, an obvious alternative site for a modified connection is indicated between points of minimum relative deflection.

Circulation Pump Vibrations in a Power Plant

The vibration spectrum from a motor/circulation pump installation at a power plant showed excessive vibration at the motor rotational frequency. The ODS was measured and characterized as a "1st plate torsional mode" of the foundation, with heavy horizontal rocking of the piping.

The solution to the problem was found after supplementary measurements of a few FRFs between the points of maximum response and the motór (residual unbalance excitation). The FRFs showed that the system was operating close to a resonance, but below the modal frequency. Traditional stiffening to raise the modal frequency may not help, in the best case, and could exacerbate the problem, in the worst case.

The solution was simply to cut two pipe supports. This shifted the modal frequency down by 20% and reduced the vibration by a factor of 10. Figure 10 shows the measured ODS and indicates where the stiffeners were cut. The photograph shows a FRF measurement at the machinery foundation for verification of the modal frequencies.



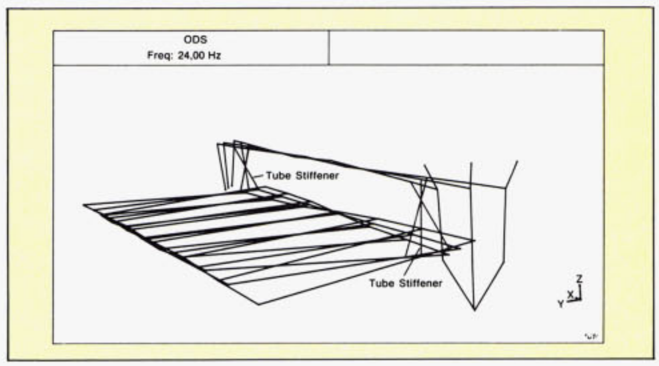


Figure 10. ODS of the main structure of a circulation pump installation at the rotational frequency of 24 Hz. The ODS appears as a vertical torsional of the foundation and a horizontal rotation of the main pipe. The solution to the problem was to cut the pipe supports, which shifted a troublesome resonance frequency down. This reduced the vibration by a factor of 10.

Ship Structure Vibrations Under Service Conditions

The ship, a 385,000 ton supertanker, had predominant vibrations at the propeller blade-passing frequency. A set of transmissibility measurements was made under different conditions, with the ship running at steady course and speed. Random variation in the excitation mechanism introduced fluctuations in the response and the measurement of reliable transmissibility values required long averaging times.

Figure 11 shows an ODS for a specific condition. The ellipses in the plot represent the orbit of the measurement point in the plane, the flag at each orbit curve represents a phase reference. Due to the complexity of a ship structure with numerous coupled modes, the ODSs are complex. The set of ODS plots serves as an interpolation map showing the global vibrations in the main structure for characteristic conditions.

Piston Compressor Vibrations

In a natural gas processing plant on board an offshore platform, excessive vibrations were causing operating problems to some other electromechanical machinery. The source was identified to be a set of two one-cylinder piston compressors on one of the lower decks. Tests showed that the problem was present when either one or both of the compressors was running. A spectrum analysis showed that all the significant vibration power was present at the 2nd harmonic of the rotational speed, which can be expected due to the free forces in this kind of machinery.

As a remedy, the first suggestion was to stiffen the deck structure. The design department calculated a construction modification, adding 10 tons of welded steel girders. However, it was suggested that some additional measurements should be made, and the ODS was established for the 2nd harmonic of the rotational frequency. The ODS showed a clean deformation shape where the two tall compressors were rocking in counter phase.

The optimal solution now became obvious: a stiffener mounted between the tops of the two compressors, a construction that weighed approximately 33 lbs. Figure 12 shows the machinery arrangement, the ODS and the alternative remedy proposals.

Conclusions

The Operational Deflection Shape designates the periodic motion pattern by which a structure vibrates at a specific frequency, under a specific stationary operating condition. In practice we often find that the spectrum, from a stationary process, exhibits one or a few predominant frequency lines or bands. Visualization of the vibration behavior at the dominant frequencies using ODS may guide the engineer to understand a problem, and provide direction to the point of a structure at which to solve a noise/vibration problem.

An Operational Deflection Shape is an observation or visualization of particular dynamic behavior, and gives no indication

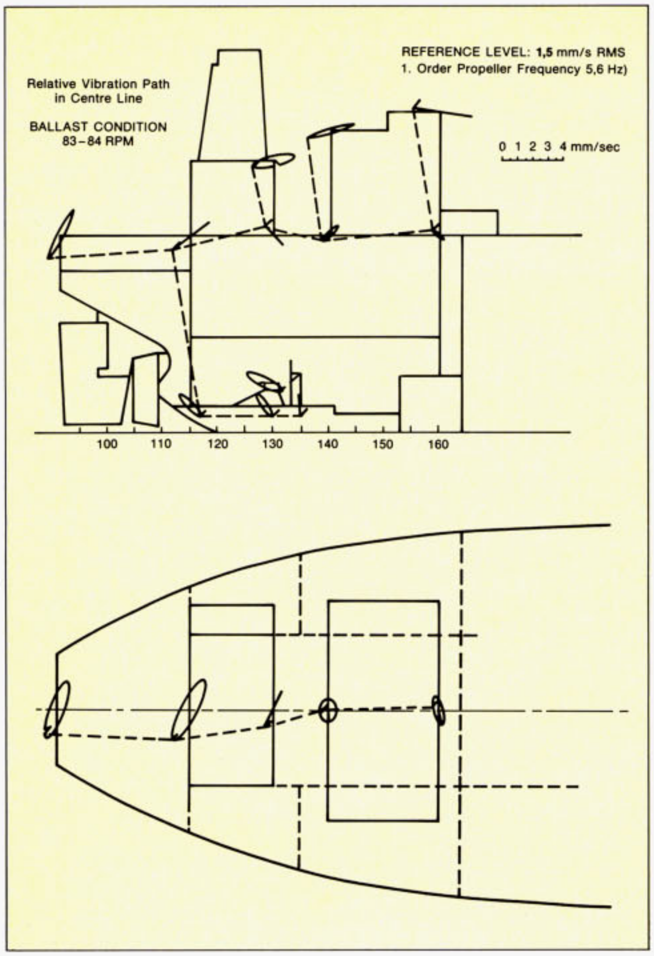


Figure 11. ODS of the main structures for a 385,000 ton super tanker in steady state ballast condition. The ellipses represent the orbits of the measured points and the flag in the path indicates equal phase. The ODS represents an interpolation map for the global vibration level.

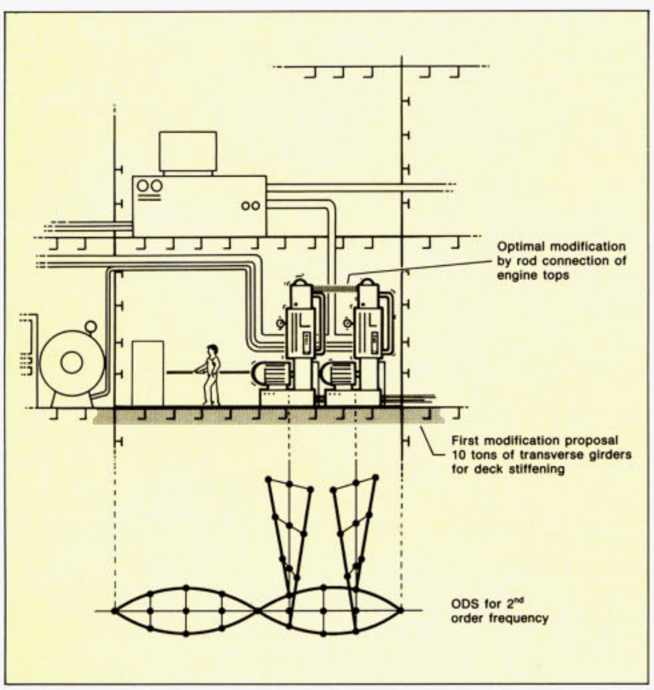


Figure 12. Double piston compessors at a gas processing plant. The ODS of 2 times the rotational frequency shows the two compressors rocking in antiphase. The optimal solution to the problem was connecting the tops of the structures.

of the qualitative dynamic properties. In other words, no model is obtained and hence no extrapolation can be made to other operating conditions. ODSs can be predicted from a mathematical model (modal model), assumed boundary conditions and, assuming the operating forces if all of these are available and the structure is linear.

If the objective is to study a particular structure under one or a few specific conditions, a direct measurement is faster, simpler and more accurate than analytical prediction, and makes no assumptions about the system. In the practical measurement the observation parameter is the (complex) transmissibility (function) representing the relative deflections (and phase), defined in points and directions over the structure. The transmissibility reveals the relative deflection between two DOFs as a function of frequency. It depends on the forcing distribution and hence, does not represent a system property.

The coherence of a transmissibility measurement tells only about the actual measurement signals. It tells nothing about the system properties, e.g. nonlinear effects. It can be used as a check of steady operating condition during the measurement, and against gross measurement errors.

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