# Is It a Mode Shape, or an Operating Deflection Shape?

**Mark H. Richardson**, Vibrant Technology, Inc., Jamestown, California

**Mode shapes and operating "deflection" shapes are related to one another. In fact, one is always measured in order to obtain the other. Yet, they are quite different from one another in a number of ways. This article discusses the relationships between modal testing, modal analysis and operating deflection shape measurements.**

The question, "Is it a mode shape, or an operating deflection shape?" is probably asked more often than any other when testing structures, especially when attempting to identify their resonant or modal properties. Another way that it is asked is, "When the excitation changes, the mode shape changes. What's going on here?"

The subject of mode shapes versus operating deflection shapes has certainly been written about before. In fact, a previous *Sound and Vibration* magazine article[1] covered them quite extensively. I recommend that you read that article, because it provides valuable insight and contains a number of examples. To shed more light on this subject, I will point out other similarities and differences between the two types of shapes, and discuss the measurements required to obtain each of them.

Over the past 20 years, the number of ways in which modal testing has been done has proliferated greatly. Traditionally, most modal testing was done using sine wave based methods and analog instrumentation. During the late 1960s however, the discovery of the Fast Fourier Transform (FFT) algorithm and the use of digital computers in laboratory testing systems allowed experimentalists to begin exploring the use of new excitation and signal processing techniques for modal testing.

Because the FFT provides the frequency spectrum of a signal in fractions of a second, various kinds of broad band random, swept sine, and transient signals, which excite many frequencies at once, could be used to excite structures and measure their responses. Impact testing has become the most popular modal testing method today. It can be done rather quickly and inexpensively using an instrumented hammer, an accelerometer, a 2 channel FFT analyzer, and post processing software. Also, the availability of lower cost transducers, PC based data acquisition systems, portable data collectors, desktop and notebook computers, and more powerful software have all helped to put modal testing into the hands of more practitioners.

Nevertheless, modal analysis has often been shrouded in a veil of mystery, while the concept of an operating deflection shape has remained relatively straightforward. Ole Døssing began his article with the statements,

> "*Operational deflection shapes (ODSs) can be measured directly by relatively simple means. They provide very useful information for understanding and evaluating the absolute dynamic behavior of a machine, component or an entire structure.*"[1]

This suggests that maybe mode shapes are not so easy to measure. If not, then why not.

## What Are Modes?

Modes are associated with structural resonances. The majority of structures can be made to resonate. That is, under the proper conditions, a structure can be made to vibrate with excessive, sustained motion. Striking a bell with a hammer causes it to resonate. Striking a sandbag, however, will not cause it to resonate.

Resonant vibration is caused by an interaction between the inertial and elastic properties of the materials within a structure. Furthermore, resonant vibration is the cause of, or at least a contributing factor to, many of the vibration related problems that occur in structures and operating machinery. These problems include failure to maintain tolerances, noisy operation, uncontrollability, material failure, premature fatigue, and shortened product life.

To better understand a structural vibration problem, we need to characterize the resonances of a structure. A common and useful way of doing this is to define its modes of vibration. Each mode is defined by a modal frequency, modal damping, and a mode shape.

Can we define modes experimentally by measuring operating deflection shapes, which are easy to measure? To answer this question requires a better understanding of mode shapes and operating deflection shapes.

## What Are Operating Deflection Shapes?

Traditionally, an ODS has been defined as the deflection of a structure at a particular frequency. However, an ODS can be defined more generally as *any forced motion of two or more points* on a structure. Specifying the motion of two or more points defines a shape. Stated differently, a shape is the motion of one point relative to all others. Motion is a vector quantity, which means that it has location and direction associated with it. This is also called a Degree Of Freedom, or DOF.

An ODS can be defined from any forced motion, either at a *moment in time*, or at a *specific frequency*. Hence, an ODS can be obtained from different types of time domain responses, be they random, impulsive, or sinusoidal. An ODS can also be obtained from many different types of frequency domain measurements, including linear spectra (FFTs), cross- and auto-power spectra, FRFs (Frequency Response Functions), transmissibilities, and a special type of measurement called an *ODS FRF*.

## Mode Shapes and ODS's Contrasted

Modes are inherent properties of a structure. They don't depend on the forces or loads acting on the structure. Modes will change if the material properties (mass, stiffness, damping properties), or boundary conditions (mountings) of the structure change. Mode shapes don't have unique values, and hence don't have units associated with them. However, *mode shapes are unique*. That is, the motion of one point relative to another at resonance is unique. We will see later that all of these conclusions can be drawn from the mathematical definition of a mode of vibration.

ODSs are quite different from mode shapes. They depend on the forces or loads applied to a structure. They will change if the load changes. ODSs can have units, typically displacement, velocity, or acceleration, or perhaps displacement per unit of excitation force. They can be used to answer the question, *"How much is the structure really moving, at a particular time or frequency?"* Finally, ODSs can be defined for nonlinear and nonstationary structural motion, while mode shapes are only defined for linear, stationary motion. ODSs can also be defined for structures that don't resonate. Modes are only used to char-

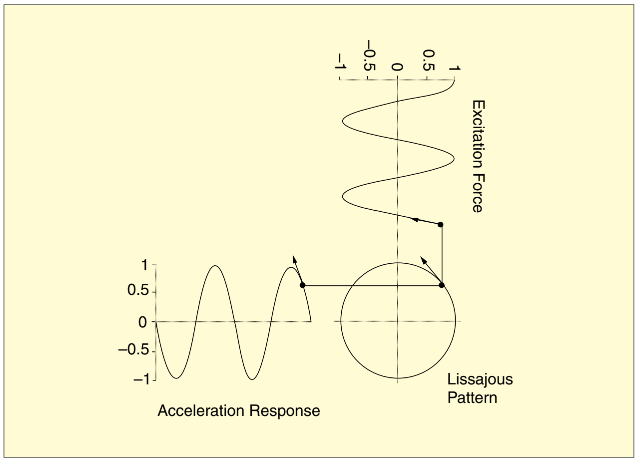

*Figure 1. Lissajous patterns indicating a pure mode.*

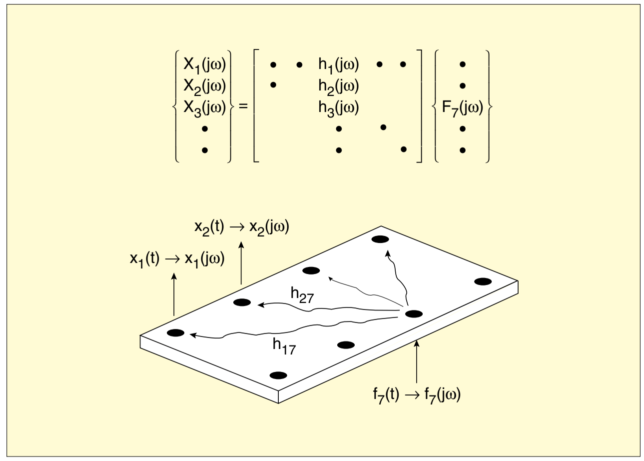

*Figure 2. FRF measurements on a structure.*

acterize resonant vibration.

## Analytical and Experimental Modes

Modes are a mathematical concept or construct, and are a convenient way of describing resonant vibration. Modes can be determined both analytically and experimentally. Analytically speaking, modes are solutions to differential equations of motion that describe the linear, stationary vibration of a structure. Experimentally, *all modal testing is done by measuring ODSs, and then interpreting or post-processing them in a specific manner to define mode shapes.*

## Traditional Modal Testing

Modal testing has traditionally been done using sine wave excitation. The first vibration test that I performed was with an eccentric shake table and a strobe light. The out-of-balance rotating mass of the shake table caused it to shake whatever was mounted on it with a sinusoidal motion. The strobe light was triggered by a tachometer signal from the shaker table, so that it illuminated the test structure with flashes of light at the same frequency as the rotational speed of the shaker. Testing was done in a room with the lights off, so that the test article could be clearly seen. The strobe light made the vibrating structure *stand still* for a brief moment of time, so that you could view its shape.

The first structure that I tested was an electronic instrument card cage. The card cage, full of printed circuit boards, was mounted securely on the shaker table. To perform a test, you merely turned on the shaker and viewed the motion of the card cage with the strobe light. As the rotation of the shaker motor was increased to higher speeds, or frequencies since its motion was cyclic, the PC boards in the cage began to exhibit excessive motions, one by one. That is, at a specific speed of the shaker motor, one board would vibrate (resonate) more than the others. At a higher speed, another board would resonate. At the resonant frequency of each board, I thought I was observing its mode shape. In fact, I was looking at its ODS.

## Normal Mode Testing

Also early in my career, I observed a normal mode test of a space satellite, at the multimillion dollar testing facility of a southern California aerospace company. The satellite had hundreds of triaxial accelerometers attached to it, each one carefully mounted so that it measured acceleration in three directions (X, Y, & Z). Several electrodynamic shakers were also attached to the satellite at different points, and in different directions.

The normal mode testing system consisted of several large bays of equipment with hundreds of small oscilloscope-like displays in them. (I was told that the system could only be operated effectively by the person who designed it.) In the language of normal mode testing, the operator used the system to *tune the modes* of the satellite.

To tune a mode, he first had to find one of the structure's resonant frequencies. This was done by merely varying the frequency of the sine wave excitation signals until the response of the satellite peaked, indicating excessive motion. Then, he adjusted the amplitudes and phases of sine wave signals sent to the shakers until all (or as many as possible) of the oscilloscopes displayed patterns that looked like circles.

These circular patterns are called Lissajous patterns. A typical Lissajous pattern display is shown in Figure 1. Each Lissajous pattern of a structural resonance is a plot of an acceleration response signal on the vertical axis versus a shaker force signal on the horizontal axis. At the frequency of a structural resonance, the structure's acceleration response is (or should be) 90° out of phase with the sinusoidal excitation force. Therefore, a plot of acceleration versus force will trace out a "circle," (actually an elipse with exactly vertical and horizontal principal axes) just like plotting a sine wave versus a cosine wave.

When the Lissajous patterns of all responses look like circles, the ODS of the structure is undergoing perfect sinusoidal motion in unison with the sinusoidal forces from the shakers. When a mode is tuned, it is called a *pure mode*, and the structure's ODS (the measured accelerations) is assumed to be its mode shape. Is this valid or not? We will look at the definition of a mode in more detail in order to answer this question.

## Modern Modal Testing Methods

Soon after the discovery of the FFT algorithm in the late 1960s, it was implemented in computer-based laboratory test instruments called FFT or Fourier analyzers. Not too long after that, in the early 1970s, a variety of new modal testing methods based on the use of the Fourier analyzer were developed.

The FFT algorithm computes a discretized (sampled) version of the frequency spectrum of a sampled time signal.[4] This discretized, finite length spectrum is called a Discrete Fourier Transform (DFT). The discovery of the FFT opened up a whole new area of signal processing using a digital computer.

## The FRF Measurement

A fundamental measurement of any multichannel FFT based data acquisition system or analyzer is the *tri-spectrum average*. Tri-spectrum averaging can be done on two or more signals that have been *simultaneously sampled*. In this averaging process, three or more spectrum estimates are computed from the signals; the *auto power spectrum* of each signal, and the *cross power spectrum* between each pair of signals. A tri-spectrum averaging loop is shown in Figure 9.

One of the key uses of the tri-spectrum average is the calculation of the Frequency Response Function (FRF), or transfer function. The FRF describes the input-output relationship between two points on a structure as a function of frequency. That is, the FRF is a measure of how much displacement, velocity, or acceleration response a structure has at an *output* point, per unit of excitation force at an *input* point.

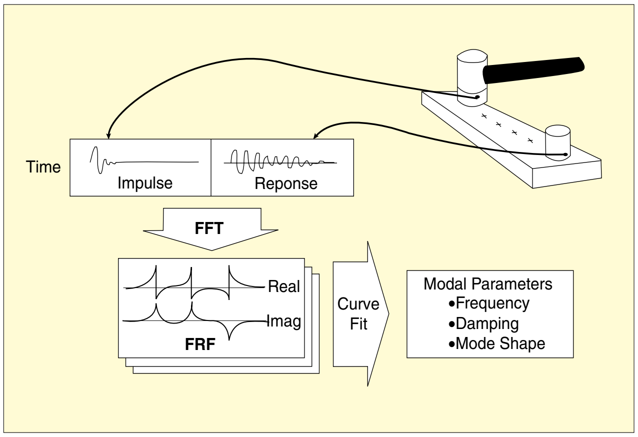


*Figure 3. Impact testing.*

The FRF is computed by dividing the cross power spectrum estimate between input and output by the input auto power spectrum estimate,

$$\text{FRF} = H(j\omega) = \frac{S_{x,f}(j\omega)}{S_{f,f}(j\omega)} \quad (1)$$

where: $S_{x,f}\ (j\omega)$ = average cross power spectrum between output and input
$S_{f,f}\ (j\omega)$ = average auto power spectrum of the input.
$j\omega$ = frequency variable

(This is, in fact, a least squared error estimate of the FRF in the presence of noise on the output signal. Other types of FRF estimators have also been defined using different noise models, but this is the most popular one implemented in FFT analyzers.) With FRF measurements, rather than excite a structure one frequency at a time with a sine wave, the structure can be excited at many frequencies, using a *broad band* signal. Broad band signals include impulses, random signals, and rapidly swept sine signals (chirps).

Each FRF measurement is computed between a sampled input signal and a sampled output signal. To obtain the mode shapes for a structure, a minimum set of FRF measurements must be taken either between a single (fixed) input and many outputs, or between a single (fixed) output and many inputs. Figure 2 depicts the FRF measurement process for an input fixed at point 7. This corresponds to measuring the FRFs in a *column* of the matrix of possible FRF measurements.

## Impact Testing

With the ability to compute FRF measurements in an FFT analyzer, impact testing became popular during the late 1970s as a fast, convenient, and relatively low cost way of finding the mode shapes of a structure. To perform an impact test, all that is needed is an impact hammer with a load cell attached to its head to measure the input force, a single accelerometer to measure the response at a single fixed point, a two channel FFT analyzer to compute FRFs, and post processing software for identifying and displaying the mode shapes in animation.

In a typical impact test, the accelerometer is attached to a single point on the structure, and the hammer is used to impact it at as many points and as many directions as required to define its mode shapes. FRFs are computed one at a time, between each impact point and the fixed response point. Modal parameters are defined by *curve fitting* the resulting set of FRFs. Figure 3 depicts the impact testing process.

## Curve Fitting

In general, curve fitting is a process of matching an analytical function or mathematical expression to some empirical data. This is commonly done by minimizing the squared error (or difference) between the function values and the data. In statistics, fitting a straight line through empirical data is called regression analysis. This is a form of curve fitting.

Estimates of modal parameters are obtained by curve fitting FRF data. Figure 4 depicts the three most commonly used curve fitting methods used to obtain modal parameters. The frequency of a *resonance peak* in the FRF is taken as the modal frequency. This peak should appear at the same frequency in every FRF measurement. The *width of the resonance peak* is a measure of modal damping. The resonance peak width should also be the same for all FRF measurements. The *peak values of the imaginary part* of the FRFs are taken as the mode shape, for displacement or acceleration responses. (The peak values of the real part are used for velocity responses.) All of these very simple curve fitting methods are based on an analytical expression for the FRF, written in terms of modal parameters.[3]

## Other Advances

Advanced measurement and post processing techniques were also developed during the 1970s and 80s. Special types of multi-shaker random and swept sine testing were developed for making better FRF measurements. These MIMO (Multi-Input Multi-Output) techniques employ specially synthesized signals to drive multiple shakers in unison with the acquisition of multiple response signals, and use matrix methods to compute FRF estimates from tri-spectrum averages.

Using multiple shakers can help insure that all resonances in a frequency band are excited, and therefore are defined in the FRFs. In addition to MIMO measurements, new curve fitting algorithms have been developed for extracting modal parameters from multiple reference (poly reference) FRFs.[6,7]

The International Modal Analysis Conference (IMAC), an annual meeting of modal testing practitioners and testing equipment vendors, began in 1980, and is still held every year. During IMAC each year, in excess of 200 technical papers are presented on many new and developing areas of modal testing.

But, what is different about modern FRF based modal testing methods versus traditional sine wave based or normal mode methods? Why are FRF measurements necessary to define modes? How is an FRF related to an ODS? To answer these questions, we'll have to dig deeper into the background math.

## Linear Dynamics

Modes are commonly defined as solutions of the following linear differential equations of motion,*

$$[\mathbf{M}]\{\ddot{\mathbf{x}}(\mathbf{t})\} + [\mathbf{C}]\{\dot{\mathbf{x}}(\mathbf{t})\} + [\mathbf{K}]\{\mathbf{x}(\mathbf{t})\} = \{0\} \quad (2)$$

where: $[\mathbf{M}]$, $[\mathbf{C}]$, and $[\mathbf{K}]$ are matrices with constants in them
$\{\ddot{\mathbf{x}}(\mathbf{t})\}$, $\{\dot{\mathbf{x}}(\mathbf{t})\}$, and $\{\mathbf{x}(\mathbf{t})\}$ are vectors of acceleration, velocity, and displacement respectively, as functions of time

This equation is, of course, a statement of Newton's Second Law; a *force balance* among the three types of internal forces in any structure made out of elastic materials. These internal forces are the inertial (mass), dissipative (damping), and restoring (stiffness) forces. Inertial and restoring forces are sufficient to cause resonant vibration. However, some form of damping is always present in all real structures, if none other than the viscous damping caused by displacement of the air surrounding the vibrating structure.

## Trapped Energy Principle

One of the most useful ways to understand resonant vibration is with the trapped energy principle. When energy enters a structure due to dynamic loading of any kind, resonant vibration occurs when the energy becomes trapped within the structural boundaries, travels freely within those boundaries, and cannot readily escape. This trapped energy is manifested in the form of traveling waves of deformation that also have a characteristic frequency associated with them. Waves traveling within the structure, being reflected off of its boundaries, *sum together* to form a *standing wave* of deformation. This stand-

*Modes can also be defined as solutions to a partial differential equation called the wave equation. The wave equation is restricted to structures with simple geometries and homogeneous properties, though.

ing wave is called a mode shape, and its frequency is a resonant or *natural* frequency of the structure.

Another way of saying this is that structures *readily absorb* energy at their resonant frequencies, and retain this energy in the form of a deformation wave called a mode shape. They are said to be *compliant* with external loads at those frequencies.

Why, then, won't a sandbag resonate when it is struck with a hammer? Energy cannot travel freely within its boundaries. The sand particles don't transmit energy efficiently enough between themselves in order to produce standing waves of deformation. Nevertheless, a sandbag can still be made to vibrate. Simply shaking it with a sinusoidal force will cause it to vibrate. Sandbags can have operating deflection shapes, but don't have resonances or mode shapes.

## Local Modes

Energy can also become trapped in local regions of a structure, and cannot readily travel beyond the boundaries of those regions. In the case of the instrument card cage described earlier, at a resonant frequency of one of its PC cards, energy becomes trapped within the card, causing it to resonate. The surrounding card cage is not compliant enough at the resonant frequency of the card to absorb energy, so the energy is reflected back and stays trapped within the card. The card vibrates but the cage does not.

Many structures have local modes; that is, resonances that are confined to local regions of the structure. Local modes will occur whenever part of the structure is compliant with the energy at a particular frequency, but other parts are not.

How do we know structures behave this way? Experimental observation is certainly one way. Another way is to solve the equations of motion, and examine the resulting modal parameters.

## Solving the Equations of Motion

To solve differential Equations (2) it is easier to transform them to the frequency domain where we can manipulate them using algebra. Using the Fourier transform, Equations (2) can be rewritten,

$$[\mathbf{B}(j\omega)]\,\{\mathbf{X}(j\omega)\} = \{\mathbf{0}\} \qquad (3)$$

where: $[\mathbf{B}(j\omega)] = [[\mathbf{M}](j\omega)^2 + [\mathbf{C}](j\omega) + [\mathbf{K}]]$ is called the *system matrix*

$\{\mathbf{X}(j\omega)\}$ is a vector of Fourier transforms of displacements

The nontrivial solution to Equation (3) is a unique set of complex *eigenvalues* and *eigenvectors*. (The trivial solution is $\{\mathbf{X}(j\omega)\} = \{\mathbf{0}\}$.) The eigenvalues occur in complex conjugate pairs. The real part of each eigenvalue is the modal damping, The imaginary part is the modal frequency. Each eigenvalue has an eigenvector associated with it. Each eigenvector is a mode shape. Each mode, then, is defined by a complex conjugate pair of eigenvalues, and a complex conjugate pair of eigenvectors, or mode shapes.

What's really important here is not the mathematical details, which can be found in a variety of references,[2,3] but the conceptual conclusions we can make regarding modes. First of all, modes are unique and inherent to any structure, the dynamics of which can be adequately described by Equations (2) or (3). Secondly, no external loads or forces are required to define modes, that is, solve Equation (3). Thirdly, modes will only change if the mass, damping, or stiffness properties of the structure are changed. Changes in boundary conditions are also reflected by changes in the **[M]**, **[C]**, and **[K]** matrices, so modes will also change if the boundary conditions change.

## How Many Equations Are Required?

Notice that Equations (2) and (3) were written with no dimensions on them. That is, the equations were written using matrices and vectors, but how large must they be? How many elements do they have in them? Or stated differently, how many equations are required? There are two fundamentally different

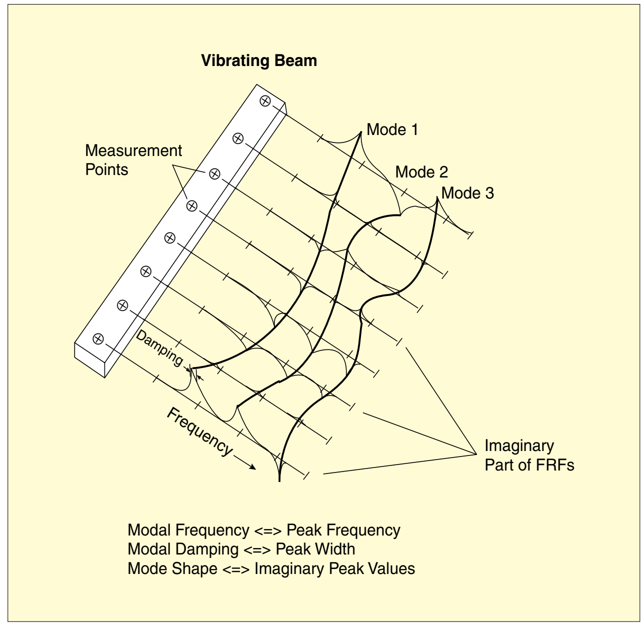


*Figure 4. Curve fitting FRF measurements.*

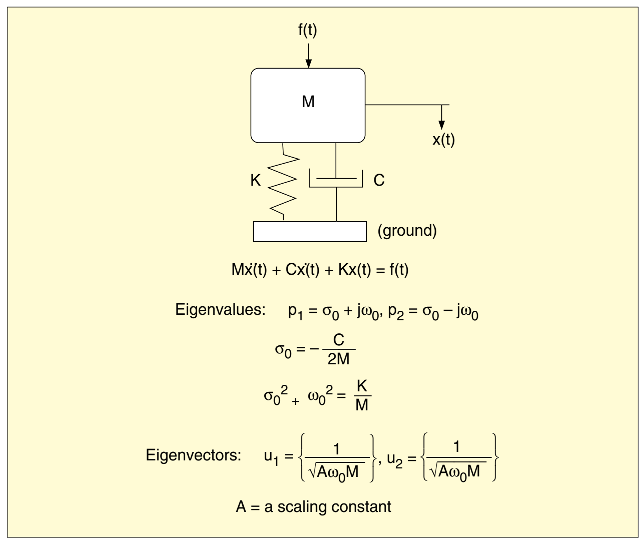


*Figure 5. Mass/spring/damper SDOF vibrator.*

ways of interpreting Equations (2) and (3); one that applies to lumped parameter and finite element models, and the other that applies to real structures.

## Single Degree Of Freedom Vibrator

To determine how many equations are required, let's start with the simplest resonant structure of all, a simple mass attached to ground with a spring and damper, shown in Figure 5. This structure is called a *lumped parameter* system because its physical properties are "lumped" into the mass, spring, and damper elements. The dynamics of this structure can be described with one differential equation, in which the mass, damping, and stiffness matrices are simply scalars. The mass matrix **[M]** is simply the mass value, the damping matrix **[C]** is the damper coefficient, and the stiffness matrix **[K]** is the spring coefficient.

Solving Equation (3) for its modes will yield a pair of eigenvalues and a pair of eigenvectors. These two eigenvalues and two eigenvectors represent a single mode of vibration. (The shapes are shown with only one DOF in them. The ground, with no motion, is the second component of the shapes.) So, a mass on a spring has only one differential equation of motion, which can be solved for one mode of vibration.

What about more complex structures like an instrument card cage or a satellite? Obviously, it will take a lot of equations to represent them dynamically. These multiple equations will, in turn, yield multiple modes as solutions.

## Testing Real Structures

Real continuous structures have an *infinite* number of DOFs, and an *infinite number of modes*. So, when we test a real structure, are Equations (2) and (3) still valid for defining its dynamics and modes? The answer is yes, if we assume that the matrices and vectors in them are still discrete, but *infinitely dimensional*. The mass, damping, and stiffness matrices, and the acceleration, velocity, and displacement vectors still have discrete elements in them, but they are infinite in dimension. That is, the number of equations we could write to describe the dynamics of a real structure is unbounded.

From a testing point of view, a real structure can be *sampled spatially* at as many DOFs as we like. There is no limit to the number of unique DOFs between which we can make measurements. The resulting set of Equations (2), (if we could compute them), would then yield an infinite set of modes, which is what real structures have. Experimental modes are actually found from a different form of the dynamic equations, described later.

Nevertheless, in testing we assume a mathematical model that is discrete, but infinitely dimensional. For practical reasons, though, we only measure a small subset of its elements. Yet, from this small subset of measurements, we can accurately define the resonances that are within the frequency range of the measurements. Of course, the more we spatially sample the surface of the structure by taking more measurements, the more definition we will give to its mode shapes.

ODSs or mode shapes of a structure are either derived directly from acquired time domain signals or from frequency domain functions that are computed from acquired time signals. We have already discussed how FRF measurements are computed.

## Forced Response

In order to make a structure vibrate, force has to be applied to it. The forced linear vibration of an elastic structure is represented by,

$$[\mathbf{M}]\{\ddot{\mathbf{x}}(\mathbf{t})\} + [\mathbf{C}]\{\dot{\mathbf{x}}(\mathbf{t})\} + [\mathbf{K}]\{\mathbf{x}(\mathbf{t})\} = \{\mathbf{f}(\mathbf{t})\} \qquad (4)$$

where: $\{\mathbf{f}(\mathbf{t})\}$ is a vector of the external forces, or loads.

*Solutions to Equation (4) are ODSs*. Recall that the definition of an ODS is the forced vibration of two or more points on a structure.

## FRF Matrix Model

An equivalent frequency domain form of the dynamic model for a structure can be represented in terms of Fourier transforms as,

$$\{\mathbf{X}(\mathbf{j}\boldsymbol{\omega})\} = [\mathbf{H}(\mathbf{j}\boldsymbol{\omega})]\{\mathbf{F}(\mathbf{j}\boldsymbol{\omega})\} \qquad (5)$$

where: $[\mathbf{H}(\mathbf{j}\boldsymbol{\omega})]$ = FRF matrix

$\{\mathbf{X}(\mathbf{j}\boldsymbol{\omega})\}$ = vector of discrete Fourier transforms of displacement responses.

$\{\mathbf{F}(\mathbf{j}\boldsymbol{\omega})\}$ = vector of discrete Fourier transforms of external forces.

Just as the time domain dynamic model is discrete but infinite dimensional, this frequency domain model is also discrete but infinite dimensional. Equation (5) is valid for all values of frequency ($j\omega$), which includes all discrete frequency values for which discrete Fourier transforms (DFTs) are computed.

Again, because real structures have an infinite number of DOFs, we can measure FRFs between as many DOF pairs (input-output pairs) as we like, as shown in Figure 2. Each FRF simply adds more definition to the dynamic model by adding another element to a row and column of the FRF matrix.

## Frequency Domain ODS

The ODS can now be defined as a solution to Equation (5). The frequency domain ODS is simply defined as the forced response at a specific frequency ($j\omega_0$),

$$\{\mathbf{ODS}(\mathbf{j}\boldsymbol{\omega}_0)\} = [\mathbf{H}(\mathbf{j}\boldsymbol{\omega}_0)]\{\mathbf{F}(\mathbf{j}\boldsymbol{\omega}_0)\} \qquad (6)$$

This equation says that the ODS is made up of a summation of vectors, each one equal to the Fourier transform of an excitation force times the *column* of FRFs corresponding to the excitation DOF. From this it is clear that the ODS is dependent upon applied external forces.

## Time Domain ODS

Taking the inverse FFT $(\mathrm{FFT})^{-1}$ of both sides of Equation (6) gives a definition of the time domain ODS,

$$\{\mathbf{ODS}(\mathbf{t})\} = \mathbf{FFT}^{-1}\{[\mathbf{H}(\mathbf{j}\boldsymbol{\omega})]\ \{\mathbf{F}(\mathbf{j}\boldsymbol{\omega})\}\} \qquad (7)$$

This equation yields an ODS vector for each value of time over which the $\mathrm{FFT}^{-1}$ is computed.

## Sinusoidal Response

The Fourier transform of a sine wave signal is nonzero at the frequency of the sine wave, and zero for all other frequencies. For a sine wave with an amplitude of ‘1’, Equation (6) simply says that the frequency domain ODS of a structure for single sinusoidal excitation is *the values of the FRFs from the reference (excitation) column, at the excitation frequency* ($\omega_0$),

$$\{\mathbf{ODS}(\mathbf{j}\boldsymbol{\omega}_0)\} = \{\mathbf{h}_{\mathrm{ref}}(\mathbf{j}\boldsymbol{\omega}_0)\} \qquad (8)$$

This is the classical definition of an ODS using FRF data. Similarly, Equation (7) can be used to define a time domain ODS for excitation by a sine wave of frequency ($\omega_0$),

$$\{\mathbf{ODS}(\mathbf{t})\} = \mathbf{FFT}^{-1}\{\mathbf{h}_{\mathbf{ref}}(\mathbf{j}\boldsymbol{\omega}_0)\} \qquad (9)$$

Since the vector of FRFs is only nonzero at a specific frequency and zero elsewhere, the inverse Fourier transform of these FRFs is merely a vector of sine waves, which is what we expect for a sinusoidal ODS.

## FRF Matrix in Terms of Modes

To further explore the relationship between mode shapes and ODSs, the FRF matrix can first be written in partial fraction form,

$$[\mathbf{H}(\mathbf{j}\boldsymbol{\omega})] = [\mathbf{H}_1(\mathbf{j}\boldsymbol{\omega})] + [\mathbf{H}_2(\mathbf{j}\boldsymbol{\omega})] + \cdots + [\mathbf{H}_k(\mathbf{j}\boldsymbol{\omega})] + \cdots + [\mathbf{H}_{\mathbf{modes}}(\mathbf{j}\boldsymbol{\omega})] \qquad (10)$$

where: $[\mathbf{H}_k(\mathrm{jw})] = \dfrac{\mathbf{1}}{\mathbf{2j}}\left[\dfrac{[\mathbf{R}_k]}{\mathrm{jw} - \mathbf{p}_k} - \dfrac{[\mathbf{R}_k]^*}{\mathrm{jw} - \mathbf{p}_k{}^*}\right]$

$\mathbf{p}_k = -\boldsymbol{\sigma}_k + \mathbf{j}\boldsymbol{\omega}_k$ = pole of the $k^{th}$ mode

$\boldsymbol{\sigma}_k$ = modal damping of the $k^{th}$ mode

$\boldsymbol{\omega}_k$ = modal frequency of the $k^{th}$ mode

$[\mathbf{R}_k]$ = a matrix of constants (called residues), for the $k^{th}$ mode.

* – denotes complex conjugate

*j* – denotes the imaginary operator.

Furthermore, each modal residue matrix can be written as the *outer product* of the mode shape vector with itself, multiplied by a scaling constant.

$$[\mathbf{R}_k] = \mathbf{A}_k\{\mathbf{u}_k\}\{\mathbf{u}_k\}^{\mathbf{tr}} \qquad (11)$$

where: $\mathbf{A}_k$ = a scaling constant for the $k^{th}$ mode

$\{\mathbf{u}_k\}$ = mode shape vector for the $k^{th}$ mode

tr – denotes transposed vector.

The denominators in Equation (10) are functions of frequency, and cause the peaks in an FRF. The location of each peak is dictated by each pole location $p_k$. ***Each peak in the FRF is evidence of at least one mode, or resonance***. An FRF measurement is shown as a summation of modal resonance curves in Figure 6.

## Measuring One Row or Column of FRFs

Modal testing is feasible because of the special form of the residue matrix; that is, the *outer product* of the mode shape (the mode shape multiplied by itself transposed). Every *row and column* of the residue matrix contains the same mode shape, multiplied by one of its own components. This is a fundamental result, which allows us to find the mode shapes of a structure by measuring only *one row or column* of FRFs from the

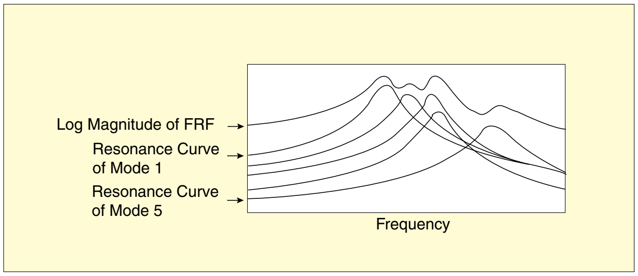


*Figure 6. FRF as a sum of resonance curves.*

FRF matrix. Without this extreme simplification, we would have to measure the entire FRF matrix instead of just one row or column. Modal testing would indeed be too time consuming!

### Curve Fitting FRFs to Obtain Modes

All forms of curve fitting that obtain modal parameters from FRF measurements use Equation (10), a simplification of it, or one of its equivalent forms as the analytical expression to be matched to the FRF data. The unknown parameters of the model are the modal parameters of the structure. The very simple curve fitting methods depicted in Figure 4 all result from simplified versions of Equation (10).[4] More sophisticated multiple mode (or MDOF) curve fitting methods are also derived from Equation (10).[7]

### ODS in Terms of Modes

By substituting the modal form of the FRF matrix (10) into the forced response Equation (6), the relationship between modes and ODSs is established.

- Every element of the FRF matrix is a summation of modal resonance curves. Therefore, the ODS contains contributions from all of the modes.
- The ODS depends not only on the excitation forces, but also on the locations of the poles (resonant peak frequencies) and the structure's mode shapes.
- If an excitation force puts energy into a structure near a resonant peak frequency, the ODS *could be very large*, depending on the value of the modal residue between the excitation and response DOFs.
- The modal residue (between an excitation DOF and a response DOF) is the *product* of the two mode shape components for the two DOFs, as shown in Equation (11). If *either* of these components is zero (a nodal line of the shape), the mode will not contribute to the ODS.

Equation (11) says what every vibration engineer knows from experience; namely, *if either the excitation or response DOF is on a nodal line of a mode shape, that mode will not contribute to the ODS*, and the mode shape cannot be obtained from any measured ODS.

### Using the Sinusoidal ODS as a Mode Shape

If the structure is excited by a single sinusoidal force, its steady state response will also be sinusoidal, regardless of the frequency of excitation. We saw this as a result in Equations (8) and (9). However, the ODS that is measured also depends on whether or not a resonance is excited.

We have already seen that in order to excite a resonance, two conditions must be met:

**Condition 1 –** The excitation force must be applied at a DOF which is *not on a nodal line* of the mode shape.

**Condition 2 –** The excitation frequency must be ***close*** to the resonance peak frequency.

If both of these conditions are met, and the resonance is "lightly" damped, it will act as a "mechanical amplifier" and greatly increase the amplitude of response, or the ODS. Conversely, if either condition is not met, the mode will not participate significantly in the ODS.

All single frequency sine wave modal testing is based upon achieving the two conditions above, plus a third:

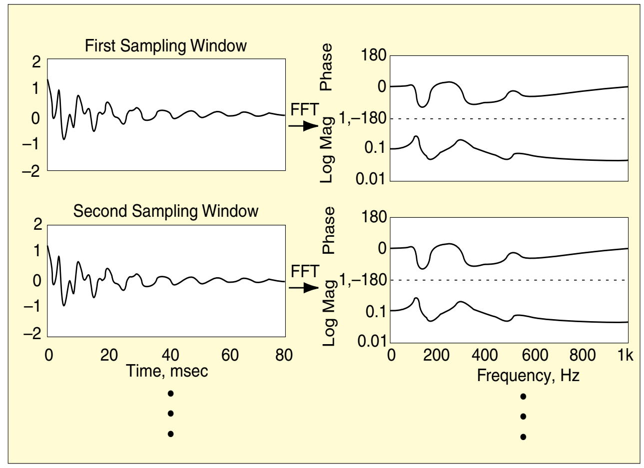


*Figure 7. Repeatable operation.*

**Condition 3 –** At a resonant frequency, if the ODS is ***dominated*** by one mode, then the ODS will closely approximate the mode shape.

Another way of saying this is that the contribution of *only one* of the terms in the summation in Equation (10) is much greater than all the rest. If Condition 3 is not met, then two or more modes are contributing significantly to the ODS, and the ODS is a linear combination of their mode shapes.

### Difficulty with FRF Measurements

We have already seen that modes can be determined experimentally by curve fitting a set of FRF measurements. As indicated by Equation (1), however, FRF measurement requires that *all of the excitation forces causing a response must be measured simultaneously with the response*. Measuring all of the excitation forces can be difficult, if not impossible in many situations. FRFs cannot be measured on operating machinery or equipment where internally generated forces, acoustic excitation, and other forms of excitation are either unmeasured or unmeasurable. On the other hand, ODSs can always be measured, not matter what forces are causing the vibration.

### Difficulty with ODS Measurements

In general, an ODS is defined with a ***magnitude*** and ***phase*** value at each point on the structure. In other words, to define an ODS vector properly, both the magnitude and *relative phase* is needed at all response points. In the frequency domain ODS given by Equation (6), this is explicit. The linear spectrum (FFT) is complex valued, with magnitude and phase at each frequency.

In the time domain ODS given by Equation (7), magnitude and phase are implicitly assumed. This means that either all of the responses have to be *measured simultaneously*, or they have to be measured under conditions which guarantee their correct magnitudes and phases relative to one another. Simultaneous measurement of all responses means that a multi-channel acquisition system, that can simultaneously sample all of the response signals, must be used. This requires lots of transducers and signal conditioning equipment, which is expensive.

### Repeatable Operation

If the structure or machine is undergoing, or can be made to undergo, repeatable operation, then response data can be acquired one channel at a time. To be repeatable, data acquisition must occur so that *exactly the same time waveform* is obtained in the sampling window, every time one is acquired. Figure 7 depicts repeatable operation. For repeatable operation, the magnitude and phase of each response signal is unique and repeatable, so ODS data can be acquired one channel at a time. This only requires a single channel data acquisition system, but an external trigger is usually required to capture the repeatable event in the sampling window. Unfortunately, repeatable operation cannot be achieved in many test situations.

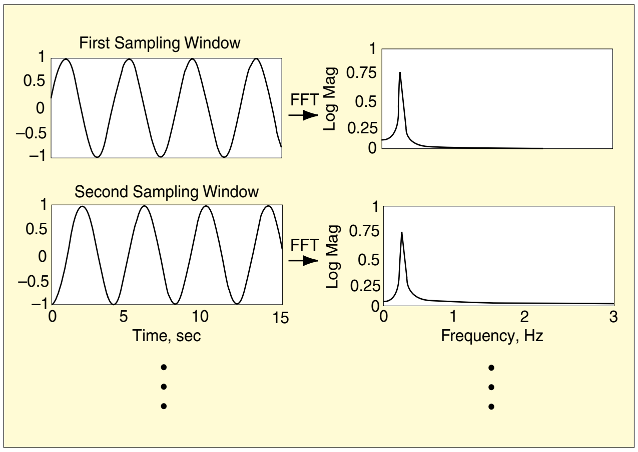


*Figure 8. Steady state operation.*

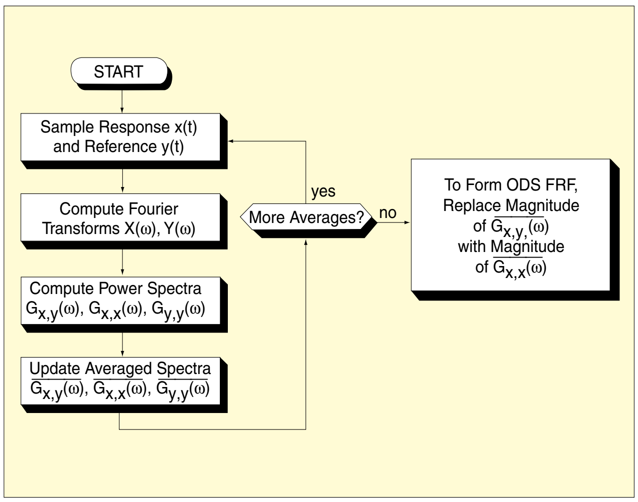


*Figure 9. Tri-spectrum averaging to compute ODS FRF.*

## Steady State Operation

Steady state operation can be achievable in many situations where repeatable operation may not be. Steady state, or stationary, operation is achieved when the *auto power spectrum* of a response signal does not change over time, or from measurement to measurement. Figure 8 shows a steady state operation. Notice that the time domain waveform can be different during each sampling window time interval, but its auto power spectrum does not change.

For steady state operation, ODS data can be measured with a 2 channel analyzer or acquisition system, that can compute a tri-spectrum average as shown in Figure 9. The cross spectrum measurement (recall that it is the numerator of the FRF), contains the ***relative phase*** between two responses, and the auto spectrum of each response contains the correct magnitude of the response. Since the 2 response signals are simultaneously acquired, the relative phase between them is always maintained. No special triggering is required for steady state operation.

## ODS FRF Measurement

Dr. Roland Angert[8] and others have suggested a new type of measurement, called ODS FRF, for obtaining ODSs from structures undergoing steady state operation. It is called an ODS FRF since it yields ODSs with correct magnitudes and phases. The real advantage of the ODS FRF measurement is that *no excitation forces need be measured*, so it can be used in situations where the input forces cannot be measured.

This new type of measurement is the result of a new measurement process, where one response signal is used as a reference, and the tri-spectrum averages are treated differently than for the standard FRF:

1. Cross spectrum measurements are made, between each response and a reference response.
2. Each ODS FRF is formed by replacing the magnitude of each cross spectrum with the auto spectrum of the response.

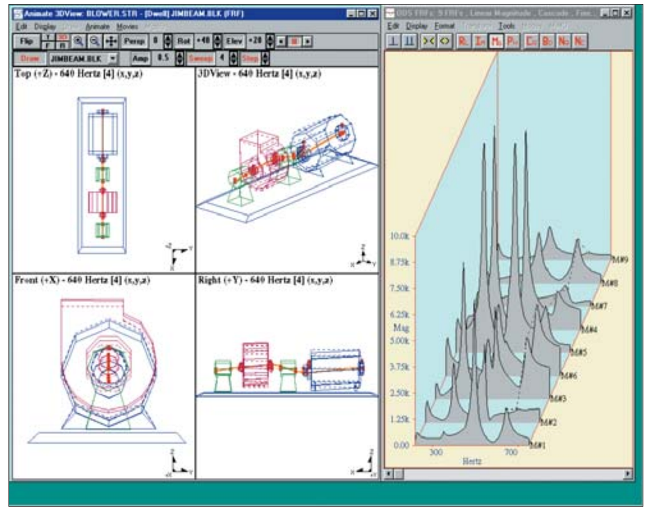

*Figure 10. Animation of ODS FRF data.*

This new measurement contains the correct magnitude of the response at each point, and the correct phase relative to the reference response. Evaluating a set of ODS FRF measurements at any frequency yields the frequency domain ODS for that frequency. Figure 10 shows an interactive display of ODSs from a set of ODS FRF measurements.

## Summary

To summarize, we have seen that mode shapes are defined quite differently from ODSs. Yet, the relationship between the two is used to obtain modes from ODS measurements.

- Modes are used to characterize resonant vibration in machinery and structures. An ODS is merely the forced response, either at a specific frequency or a specific moment in time.
- Modes are inherent properties of resonant structures. ODSs can be defined for any structure, resonant or not.
- An ODS changes with structural loading, a mode shape does not. The ODS is completely arbitrary, depending on the combination of excitation forces acting on the structure. Modes only change with changes in the physical properties (mass, stiffness, or damping) or boundary conditions of the structure.
- An ODS can answer the question, “How much is the structure actually moving?”. A mode shape has no units, so it cannot be used by itself to answer this question.
- Modes are solutions to linear, stationary differential equations of motion. Therefore, they can only be measured when a structure is undergoing linear, stationary motion. ODSs are defined for any type of motion, including nonlinear and time varying motion.
- Modes define resonances, which can be characterized as energy trapped within the boundaries of a structure. In order for a resonant standing wave motion to occur, energy must travel freely within the boundaries of the structure, and not readily escape. An ODS can be defined for nonresonating structures that don’t satisfy these conditions.
- All modal testing is done by measuring an ODS, and then applying special interpretation (e.g. for sine testing the ODS *is* the mode shape), or special post-processing (curve fitting) to obtain modal parameters.

ODSs always contain contributions due to resonances and excitation forces. We saw from looking at the analytical form of the FRF matrix model written in terms of modal parameters, that when the modes are not heavily coupled together (typical for lightly damped structures), *the ODS at or near a resonant frequency is dominated by a single mode shape.* This fact is the underlying assumption behind all normal mode testing, and is also used when single mode (SDOF) curve fitting methods are used to obtain modal parameters from FRFs.

Modes can be obtained by curve fitting FRFs, but measurement of FRFs requires that all excitation forces be measured, which is not always possible. ODS measurements can be made without measuring forces, but further assumptions are also required. The ODS FRF measurement is ideal for analyzing the ODSs of operating machinery. From ODS FRF measurements, the question "How much is the machine or structure really moving?" can be answered.

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